GRADIENT INFLUENCED SPACE CHARGE INSTABILITIES
IN MPD THRUSTERS

H.P. Wagner, M. Auweter-Kurtz, E.W. Messerschmid
Institut für Raumfahrtsysteme (IRS)

H. J. Kaeppeler
Institut für Plasmaforschung (IPF)
Universität Stuttgart, FRG

Abstract

Since a few years, the study of magnetoplasmadynamic instabilities has received increased attention as they may explain the "onset" phenomenon, which limits the efficiency and, thus, the use of MPD thrusters in space applications. In the previous part of their systematic investigation of macroscopic instabilities in MPD devices, the authors found at least one presenting the characteristics of a space charge instability. This instability was known to occur only in finite discharge geometries.

It will be proven in this report, that, in the presence of gradients of the flow variables, space charge instabilities may also appear in unbounded plasmas. The proper conditions for their build-up are determined and discussed. Finally, it will be shown, that this instability explains some of the properties of the one found in the previous works.

1 Introduction

A systematic investigation of magnetoplasmadynamic instabilities occurring in MPD thrusters was started at the Institut für Raumfahrtsysteme (IRS) in 1987. Macroscopic and/or drift instabilities are suspected to favour or even to cause the "onset" phenomenon limiting the development of highly efficient thrusters. Contrary to the work at the IRS, Kuriki, Choueiri, Princeton University and Hastings, MIT, studied mainly the occurrence of plasma microinstabilities and their effects on the thruster flow. The authors at the IRS/IPF believe that unstable behaviour causing "onset" is linked to the current transport in the plasma and therefore is influenced by macroscopic instabilities.

Thus, a macroscopic linear dispersion relation is derived from a reduced three-fluid model of the plasma flow in the thruster. This model comprises the three equations of continuity for the electrons, ions and neutrals, two equations of motion and energy for the electrons and the heavy particles (ions and neutrals), as well as Maxwell’s equations, and is therefore currently the most extensive model ever used in a normal mode stability analysis.

First results obtained by Rempfer outside the discharge region of the thruster showed the appearance of an electron acoustic wave instability. As it can be assumed from the work by Hügel that the "onset" phenomenon originates near the anode of the thruster, the following stability analysis was performed with finite values for the electric field strength \( E \), the magnetic induction \( B \) and with finite gradients in the particle density \( n \) and temperature \( T \). Up to five instabilities were found, depending on the importance assigned to various transport processes. Among these five instabilities, only two are of particular interest as they are purely gradient driven. The root loci containing these instabilities (points with minimum \( \Im(\omega) \)) are plotted for various values of the relative gradient in Fig. 1 and Fig. 2. The instabilities as well as the corresponding root loci were arbitrarily numbered Nr.2 and Nr.3 in the previous paper.

![Fig. 1: Locus of root Nr. 3](image)

Instability Nr.3 was tentatively identified as an electron acoustic wave instability. Further analytical investigation seems to show that it can convert into an unstable acoustic mode of the ions for the case of \(
91-101

![Graph](image)

**Fig. 2:** Locus of root Nr. 2

fully ionized plasma. This has to be studied in further
detail.

Instability Nr.2 is assumed to be a space charge insta-
bility with its frequency $\Re(\omega) \approx 0$. It will be shown
in this work that a space charge instability, originally
defined in a bounded plasma\(^{18}\), can also appear in an
unbounded plasma.

## 2 The Space Charge Instability

Space charge instabilities, also called Pierce or diode
instabilities, are of particular interest in discussing
charge carrier deficits and arc starvation. The sta-
bility of discharges between electrodes with fixed
potentials\(^{10, 11, 12, 13}\) were studied 50 years ago in con-
nection with current transport in diodes\(^{14, 15, 16}\). In
the 1960ies, this phenomenon was considered a pos-
sible explanation for current interruptions in $m = 0$
instabilities\(^{17}\).

Consider a discharge between two plane parallel and
infinitely extended electrodes which are a finite dis-
tance $L$ apart (Fig. 3). The discharge current is as-
sumed to be carried only by electrons, emanating with
density $n_0$ and velocity $v_0$ from the cathode and be-
ing absorbed at the anode. The equilibrium charge of
the electron stream is compensated by uniformly dis-
tributed inert ions.

Simplified equations of continuity and motion for
the electrons are used to describe the system. Fur-
thermore, as magnetic induction is neglected, the electro-
static field can be determined by Poisson's equation

$$\text{div}(E) = \frac{e}{\varepsilon_0} (n_e - n_0) .$$

Introducing the electric field potential $\varphi$ with
$\text{grad}(\varphi) = -E$, the equations used are, thus,

$$\frac{\partial n_e}{\partial t} + \text{div}(n_e \vec{v}_e) = 0 ,$$
$$\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \text{grad}) \vec{v}_e - \frac{e}{m_e} \text{grad}(\varphi) = 0 ,$$
$$\text{div}(\text{grad}(\varphi)) + \frac{e}{\varepsilon_0} (n_i - n_e) = 0 .$$

Using an ansatz for longitudinal perturbations (along
the electric field $E$)

$$n_e = n_0 + n_1(x,t) ,$$
$$\vec{v}_e = \vec{v}_0 + v_1(x,t) ,$$
$$\varphi = \varphi_0 + \varphi_1(x,t) ,$$

with $n_i$ being constant and the perturbed values $\psi_1$
being small in magnitude compared to the unperturbed
ones $\psi_0$, the system of equations (1) becomes

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial n_1}{\partial x} = 0 ,$$
$$\frac{\partial v_1}{\partial t} + v_0 \frac{\partial v_1}{\partial x} - \frac{e}{m_e} \frac{\partial \varphi_1}{\partial x} = 0 ,$$
$$\frac{\partial^2 \varphi_1}{\partial x^2} - \frac{e}{\varepsilon_0} n_1 = 0 .$$

In (2), the unperturbed values as well as $n_1$ are as-
sumed to obey (1) and products of perturbed variables
are neglected. A variable transformation according to

$$n_1 = \tilde{n}_1(x) e^{i\omega t} ,$$
$$v_1 = \tilde{v}_1(x) e^{i\omega t} ,$$
$$\varphi_1 = \tilde{\varphi}_1(x) e^{i\omega t} ,$$

is performed, corresponding to a Fourier transforma-
tion in time only. Dropping the subscript "1", this
yields for the system (2)

$$i \omega \tilde{n} + \frac{n_0}{\varepsilon_0} \frac{\partial \tilde{v}}{\partial x} + v_0 \frac{\partial \tilde{n}}{\partial x} = 0 ,$$
$$i \omega \tilde{v} + \frac{v_0}{m_e} \frac{\partial \tilde{\varphi}}{\partial x} = 0 ,$$
$$i \omega \tilde{\varphi} - \frac{e}{\varepsilon_0} \tilde{n} = 0 .$$

By repeated spatial derivation and elimination of $\tilde{n}$
and $\tilde{v}$ from (3), there follows a linear differential
equation in $\tilde{\varphi}$

$$\frac{d^4 \tilde{\varphi}}{dx^4} + 2i \omega \frac{\partial^3 \tilde{\varphi}}{\partial x^3} - \omega_w^2 \frac{\partial^2 \tilde{\varphi}}{\partial x^2} = 0 ,$$

where $\omega_p$ is the electron plasma frequency. Equation
(4) can be analytically solved and thus the general
solution of (3) is

\[ \dot{\phi} = Ae^{-i \frac{\omega_0}{v_0} x} + Be^{-i \frac{\omega_p}{v_0} x} + Cx + D, \]

\[ \dot{n} = -\frac{\varepsilon_0}{e v_0} \left[ A(\omega + \omega_p)^2 e^{-i \frac{\omega_0}{v_0} x} + B(\omega - \omega_p)^2 e^{-i \frac{\omega_p}{v_0} x} \right], \]

\[ \dot{v} = \frac{e}{m_e \omega_p v_0} \left[ A(\omega + \omega_p) e^{-i \frac{\omega_0}{v_0} x} - B(\omega - \omega_p) e^{-i \frac{\omega_p}{v_0} x} \right] - Ci \frac{e}{\omega m_e}. \]

The constants of integration are determined by boundary conditions. The potential \( \phi \) at both electrodes as well as the electron density \( n_e \) and velocity \( \vec{v}_e \) at the cathode are fixed, thus,

\[ \begin{align*}
\dot{\phi}(0) &= 0, \\
\dot{\phi}(L) &= 0, \\
\dot{n}(0) &= 0, \\
\dot{v}(0) &= 0.
\end{align*} \tag{6} \]

The conditions (6) lead to a homogenous system of equations for the unknowns \( A, B, C \) and \( D \). It has a nontrivial solution only if the determinant of its coefficient matrix is equal to zero, which determines the dispersion relation. Introducing the variables \( \xi = L \omega/v_0 \) and \( \alpha = L \omega_p/v_0 \), it yields

\[ 2 \xi \alpha (1 - e^{-i \xi} \cos \alpha) - i e^{-i \xi} (\xi^2 + \alpha^2) \sin \alpha - i \frac{\xi^2}{\alpha} (\xi^2 - \alpha^2) = 0. \tag{7} \]

The formalism given up to here follows closely that of Mikhailovskii\(^{18}\) who gives an approximate solution for equation (7):

For \( \alpha \) being slightly greater than \( \alpha_\ell \) with

\[ \alpha_\ell = \ell \pi \quad \ell = 1, 3, 5, \ldots, \]

the complex frequency \( \omega \) is approximately

\[ \Re(\omega) = 0 \quad \text{and} \quad \Im(\omega) = -\gamma \approx -\frac{v_0}{L} \], \tag{8} \]

where \( \gamma \) is the growth rate.

A more detailed numerical analysis of (7) by the authors showed the existence of an infinity of solutions periodically distributed over the complex plane and with the line \( \Re(\omega) = 0 \) being a symmetry axis for all \( \alpha \). As far as investigated, only one set of zeros corresponds to unstable solutions of the system (3), their real part obey

\[ \Re(\omega(\alpha)) = 0 \], \tag{9} \]

and their imaginary part is plotted in Fig. 4 for positive and negative \( \alpha \).

Fig. 4: Dispersion graph of the Pierce instability

The values of the plasma parameters \( \omega_p \) and \( L \) were chosen as follows\(^{1, 2}\):

\[ \omega_p = 2.9040 \cdot 10^{12} \text{ rad/m}, \]

\[ L = 5.0000 \cdot 10^{-5} \text{ m}. \]

The value of \( L \) is equal to the inverse relative gradient (scaling length) \( e^{-1} = 20000^{-1} \text{ m} \) and the value of \( v_0 \) was calculated with

\[ v_0 = \frac{L \omega_p}{\alpha(\ell)} i = 1, 2. \]

The results are shown in Fig. 5 for the unstable solution and in Fig. 6 for the stable solution, where the variables \( \dot{\phi}, \dot{n} \) and \( \dot{v} \) have been normalised.

The calculation of the spatial distributions shows, that for an eigenvalue \( \alpha \) near \( \alpha_\ell = \ell \pi \) with \( \ell = 1, 3, 5, \ldots \), the space between the electrodes is divided in \( \ell \) regions of different thicknesses, in which the potential perturbation \( \dot{\phi} \) is of alternate sign. It has to be understood, as \( \Re(\omega) = 0 \), that the solution is not a standing wave but increases (or decreases) monotonously with time in the case of an unstable (or stable) mode.
Therefore, the stable solutions as represented in Fig. 6 tend to leave the discharge current unchanged whereas the unstable solutions build up at least one space charge sheath ($\ell = 1$) which may lead to an arc starvation in a nonlinear development.

### 3 The Influence of Gradients

It was shown in the preceding section that the build-up of a space charge instability requires the ratio $v_0/L$, introduced by the boundary conditions (6), to be finite. This ratio can be considered as a "gradient" of the velocity in the space between the electrodes. It will be shown in the following that the presence of such gradients will cause space charge instabilities to occur even in an unbounded plasma.

Considered is the discharge described in Section 2, except that the electrodes are an appreciable distance apart and do not directly determine the plasma properties. Furthermore, there exist gradients in the unperturbed variables $n_0$ and $v_0$. The behaviour of the plasma is described by the system of equations (1).

Using the perturbation ansatz

\[
\begin{align*}
n_e &= \hat{n}_0(x) + n_1(x, t), \\
\vec{v}_e &= \hat{v}_0(x) + v_1(x, t), \\
\varphi &= \hat{\varphi}_0(x) + \varphi_1(x, t),
\end{align*}
\]

as well as a complete Fourier transformation in space and time

\[
\begin{align*}
n_1 &= \hat{n} e^{i(\omega t - k_x x)}, \\
v_1 &= \hat{v} e^{i(\omega t - k_x x)}, \\
\varphi_1 &= \hat{\varphi} e^{i(\omega t - k_x x)},
\end{align*}
\]

the system (1) becomes:

\[
\begin{align*}
\left[ \frac{\partial n_0}{\partial x} + i(\omega - k_x v_0) \right] \hat{n} + \frac{\partial \hat{n}_0}{\partial x} - i k_x n_0 \hat{v} &= 0, \\
\left[ \frac{\partial v_0}{\partial x} + i(\omega - k_x v_0) \right] \hat{v} + i k_x \frac{\epsilon_0}{m_e} \hat{\varphi} &= 0, \\
k_x^2 \hat{\varphi} + \frac{\epsilon_0}{\epsilon_0} \hat{n} &= 0.
\end{align*}
\]

Introducing the relative gradients $\epsilon_n$ and $\epsilon_v$,

\[
\epsilon_n = \frac{1}{n_0} \frac{\partial n_0}{\partial x} \quad \text{and} \quad \epsilon_v = \frac{1}{v_0} \frac{\partial v_0}{\partial x},
\]

Fig. 5: Spatial distributions for $\alpha^{(1)}$ (unstable case)

Fig. 6: Spatial distributions for $\alpha^{(2)}$ (stable case)
and computing the determinant of the coefficient matrix of (11) yields the linear dispersion relation

\[ [v_0 \epsilon_v + i(\omega - k_x v_0)]k_x + i \omega_p^2 (\epsilon_n - i k_x) = 0 \]  

(12)

In the most general case, the wave vector \( k_x \) is complex to allow the appearance of spatial instabilities. In accordance with the transformation (10), these instabilities occur only if the imaginary part of \( k_x \) is positive and they should not be confused with temporal instabilities which occur if the imaginary part of the frequency \( \omega \) is negative. With

\[ k_r = \Re(k_x), \]

\[ k_i = \Im(k_x) \]

and the abbreviation

\[ \beta = \sqrt{\frac{k_r^2 + k_i^2 + \epsilon_n k_i + \sqrt{(k_r^2 + k_i^2 + \epsilon_n k_i)^2 + \epsilon_n^2 k_i^2}}{k_r^2 + k_i^2}}, \]

equation (12) can be solved for \( \omega \):

\[ \omega^{(\pm)} = \left[ k_r v_0 \pm \frac{\omega_p (k_r \epsilon_n + \sqrt{\omega_p})}{\sqrt{2(k_r^2 + k_i^2)}} \right] + i \left[ (k_i + \epsilon_n) v_0 \pm \frac{\omega_p (k_r \epsilon_n + \sqrt{\omega_p})}{\sqrt{2(k_r^2 + k_i^2)}} \beta^{-1} \right]. \]

(13)

As seen in Section 2, one property of space charge instabilities is that \( \Re(\omega) = 0 \), therefore the condition

\[ \beta^{(\pm)} = \pm \frac{\sqrt{2k_v v_0}}{\omega_p} \]  

(14)

is inserted into \( \Im(\omega^{(\pm)}) \), using \( k = |k_x| \), to yield

\[ \Im(\omega^{(\pm)}) = v_0 \left( k_i + \epsilon_v - \frac{\omega_p^2 \epsilon_n}{2k_r^2 v_0^3} \right). \]

(15)

Thus, both solutions \( \omega^{(+) \pm} \) and \( \omega^{(-)} \) become unstable if one of the following conditions is fulfilled:

\[ v_0 > 0 \quad and \quad \epsilon_n > \frac{2k_r^2 v_0^2}{\omega_p^2} (k_i + \epsilon_v), \]  

(16)

or

\[ v_0 < 0 \quad and \quad \epsilon_n < \frac{2k_r^2 v_0^2}{\omega_p^2} (k_i + \epsilon_v). \]  

(17)

It has to be shown now that there exists at least one pair \((k_r, k_i)\) which fulfills the condition (14). Rewriting equation (14) with \( k_x = k \cos \psi + i k \sin \psi \) yields, after elimination of the radicals

\[ k^6 (\cos \psi)^2 - k^4 \frac{w_p^2}{v_0^2} - k^3 \frac{w_p^2}{v_0^2} \epsilon_n \sin \psi - \frac{\epsilon_n^2 w_p^4}{4v_0^2} = 0. \]

(18)

With the supplementary requirements of \( k \) being real and positive and \( k_r, k_i \) being real, not every solution of (18) will always meet the condition (14), depending on the values given to \( \omega_p, v_0 \) and \( \epsilon_n \). Considering purely real \( k_x \) yields an approximate solution with

\[ k_r \approx \pm \frac{\omega_p}{v_0} \quad and \quad k_i = 0, \]

(19)

which fulfills (14). On the other hand it seems that \( k_r = 0 \) does not lead to solutions \( k_i \in \mathbb{R} \).

### Table 1: Stability table

<table>
<thead>
<tr>
<th>( v_0 )</th>
<th>( \nabla v_0 )</th>
<th>( \nabla n_0 )</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>U if (</td>
</tr>
<tr>
<td>( \geq 0 )</td>
<td></td>
<td></td>
<td>U</td>
</tr>
<tr>
<td>= 0</td>
<td>&lt; 0</td>
<td></td>
<td>S</td>
</tr>
<tr>
<td>&gt; 0</td>
<td>≤ 0</td>
<td></td>
<td>indifferent</td>
</tr>
</tbody>
</table>

4 Discussion of the results

The results obtained in Section 3 show clearly that an instability appears having the same properties as the space charge instability: The space is divided into regions of thickness \( \lambda \approx \pi/k_r \), in which the perturbed variable \( n \) alternates its sign. Again, the solution is not a wave, as, with \( \Re(\omega^{(\pm)}) = 0 \), the perturbation is monotonously (aperiodically) increasing in time and spatially decreasing or increasing, depending on the value of \( k_i \). Therefore no phase or group velocities and, in consequence, no anomalous transport rates between...
plasma components can be defined. In a further evolution of the instability, the kinetic energy of the electrons will not be sufficient anymore to overcome the potential hump created by the space charge sheath, thus, the discharge current is locally interrupted. The ensuing arc starvation causes the instability to vanish and the current to return to its previous level. The process repeats and may manifest itself through strong current/voltage oscillations of large amplitude.

The role of the velocity gradient is not so important as suggested by the equations (7) and (8) in Section 2, where the instability vanishes with the “gradient of velocity” \( v_0/L \) when \( L \to \infty \). In an unbounded plasma, the gradient of velocity \( \nabla v_0 \) may become zero without preventing the space charge instability, depending on the values of \( v_0 \) and the density gradient \( \nabla n_0 \). Table 1 summarizes the stability conditions of (16) and (17) with \( k_2 = \pm \omega_p/v_0 \).

With \( k_2 = \pm \omega_p/v_0 \) and \( \epsilon_v = 0 \) the growth rates \( \gamma^\pm \) for \( \omega^\pm \) become

\[
\gamma^\pm = \frac{v_0}{2} \epsilon_n.
\]  

The loci of \( \omega^\pm \) for \( |k_t| \in [10^7, 10^9] \) rad/m, \( k_t = 0 \), \( \omega_p = 2.9040 \cdot 10^{12} \) rad/s, \( v_0 = 14.66 \) km/s, and \( \epsilon_v = 0 \) and \( \epsilon_n = 2.0 \cdot 10^4 \) 1/m are shown in Fig. 7 and Fig. 8, respectively.

The value of \( v_0 \) is the same as in Section 2 whereas the values for the relative gradients \( \epsilon_n \) and \( \epsilon_v \) are taken from the authors’ previous papers\(^1,2\).

The results discussed above have to be compared to those of the instability Nr. 2 found previously from the three-fluid theory and which was assumed to be a space charge instability\(^1,2\) (Fig. 2). Let \( \omega^2 \) be the point with maximum growth rate \( \gamma^2 \) of those root loci and \( k_2^2 \) its wave vector. Table 2 summarizes the characteristic data of the instabilities \( \omega^2 \) and \( \omega^- \) with the plasma parameters \( \omega_p = 2.9040 \cdot 10^{12} \) rad/s, \( v_0 = 560 \) m/s and \( \epsilon_v = 0 \).

It can be seen that equation (20) can explain the linear dependence of \( \gamma^2 \) on \( \epsilon_n \) and the numerical values of the growth rates agree fairly well. On the other hand, if the finite values of \( \Re(\omega^2) \) can still be considered to be zero in regard to the plasma frequency \( \omega_p \), the values of \( k_2^- \) being five orders of magnitude larger than \( k_2^2 \) cannot be ignored. Furthermore, as suggested in Fig. 2,

\[
\lim_{k_2 \to 0} \Re(\omega) = 0.
\]

Obviously, the much more complicated three-fluid model contains dissipation effects which are neglected in the simple model treated here. It is quite likely that the dissipation effects of the three-fluid system lead to a reduction of spatial dispersion which could explain the difference in the \( k_2 \)-values. This will be investigated in more detail.

**5 Conclusions**

Using a two-fluid model of a plasma with moving electrons and inert ions, it was shown that the so-called space charge instability, assumed to occur only in plasmas limited by electrodes, can also appear in unbounded plasmas in the presence of gradients of the flow variables.

With a normal mode stability analysis, two unstable solutions were found which both are able to generate a space charge instability under the same conditions. The proper configuration of velocity and gradients for the build-up of such an instability were determined and discussed in detail. Furthermore, some of the properties of an instability found previously\(^1,2\), assumed to be a space charge instability, could be explained.

Therefore, the Pierce instability, leading to a current chopping in its nonlinear evolution, is considered a possible explanation of the “onset” phenomenon in MPD thrusters. Further investigations will include the effects of temperature gradients, moving ions and especially transport processes, in order to explain the instability properties.
6 Acknowledgements

This work is supported by the Deutsche Forschungsgemeinschaft (DFG) under grant Me951/1-2.

References


