TWO-DIMENSIONAL NUMERICAL MODEL OF PLASMA FLOW IN A HALL THRUSTER

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Abstract

Two-dimensional numerical model of plasma flow in a Hall thruster has been made to analytically estimate the ion-loss flux to the walls of an acceleration channel, and to obtain information about desirable configuration for good thruster performance. The model presented herein is comprised of an electron diffusion equation and an ion kinetic equation, which enable one to compute electrostatic potential contours and ion-beam trajectories. In the first step, ion-production distribution was assumed. From the results, it was found that electric field distortion, which is a main cause of ion loss to the channel walls, is induced not only due to the curvature of magnetic field lines, but also due to the radial non-uniformity of ion-production distribution. In the second step, the ion-production distribution was self-consistently determined by combining an energy conservation equation with the above two basic equations. The results indicate that the profile of ion-production distribution largely changes with the magnetic field geometry, and hence the field geometry significantly influences the ion-loss flux to the channel walls. The computed ion-loss fraction (a fraction of ions produced that are lost to the walls) and profiles of plasma properties show good agreements with measured ones, and therefore, this model should be an effective tool in both the design and improvement of Hall thrusters.

Nomenclature

- $B$ = magnetic induction
- $D$ = electron diffusion coefficient
- $E$ = electric field
- $e$ = electronic charge
- $F$ = thrust
- $f$ = beam-ion energy distribution function
- $I_a$ = acceleration current
- $I_b$ = ion-beam current
- $I_e$ = electron current
- $I_p$ = ion-production current
- $J_e$ = electron current density
- $J_i$ = ion current density
- $k$ = Boltzmann constant
- $M$ = ion mass
- $m$ = electron mass
- $m_p$ = propellant mass flow rate
- $n$ = plasma density
- $q$ = ion-production rate
- $T_e$ = electron temperature
- $t$ = time
- $V$ = volume of an element
- $V_a$ = acceleration voltage
- $V_m$ = mean beam-ion energy
- $v$ = ion velocity
- $x$ = ion position
- $\alpha$ = ion-loss fraction
- $\beta$ = ion-production coefficient
- $\Gamma_i$ = ion flux
- $\gamma$ = excitation coefficient
- $e$ = ionization energy
- $\kappa$ = heat conductivity
- $\phi$ = space potential
- $\mu$ = electron mobility
- $\eta_a$ = acceleration efficiency
- $\eta_b$ = beam energy efficiency
- $\eta_t$ = thrust efficiency
- $\eta_u$ = propellant utilization
- $v$ = electron collision frequency
- $\theta$ = angle of field lines respect to the axis

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Introduction

Hall thrusters have axisymmetric electrodes and acceleration channel in which radial magnetic fields are applied to maintain a relatively high voltage between anode and cathode neutralizer, and ions generated in the channel are accelerated in the axial direction to produce thrust. A schematic of a Hall thruster is shown in Fig. 1. Since the channel is filled with quasi-neutral plasma, there is no space-charge limited current, and hence this type of thruster can offer much higher thrust density than conventional ion thrusters. With this characteristic, there
has been a trend towards re-evaluating Hall thrusters in the United States\textsuperscript{1,2} and Japan\textsuperscript{3} since 1980's. On the other hand, Russia maintained a substantial research and development effort over 30 years, and more than 50 of these thrusters have been utilized in space for station keeping of satellites.\textsuperscript{4,5}

In our previous work on Hall thrusters,\textsuperscript{6,7} thruster performance was improved through some modifications in thruster configuration; shortening the channel length together with arranging the magnetic field lines to be perpendicular to the axis. However, thrust efficiency was still low compared with that of conventional ion thrusters. From the measurement of the plasma properties inside the channel, the ion current lost to the channel walls is found to be considerably large, resulting in low thrust efficiency. In order to reduce the ion loss and to improve the thrust efficiency, analytical approach would be very effective in addition to the experimental efforts.

![Hall thruster schematic diagram](image)

**Fig. 1 Hall thruster schematic diagram.**

**Numerical Model**

On the design of Hall thrusters, the channel length is chosen to be larger than electron cyclotron radius and to be smaller than ion cyclotron radius, so that the applied magnetic field works preferentially on electrons, but not on ions. Besides, ion mean free path is usually longer than the channel length, and electron Hall parameter is much larger than unity in the channel, because the density is not so high as that in MPD thrusters. In these conditions, ions are electrostatically accelerated downward without collisions, whereas electrons are constrained to azimuthal drift motions by the interaction with the radial magnetic fields. However, axial electron current arises simultaneously from the classical or Bohm diffusion in the direction of electric field.

The ionization process in a Hall thruster depends on its operational type. In the case of single-stage discharge operation, propellant gas is injected from the anode and ionized in the channel by the discharge between anode and cathode neutralizer. On the other hand, in the case of double-stage discharge operation,\textsuperscript{3,8} a plasma source is located at the upstream end of the acceleration channel, and ions are produced there by a discharge independent of the main discharge.

**Assumptions**

This numerical model is based on the following assumptions. A steady-state, axisymmetric flows of ions and of electrons are considered. Ions are electrostatically accelerated without collisions, while electrons diffuse in the axial and the radial directions by electric fields and by density gradient in the acceleration channel. The diffusion across the magnetic field line is important for thruster operation. In case that $1/B^2$ classical diffusion is assumed in the direction perpendicular to the magnetic field, the electron diffusion current backstreaming in the channel becomes orders of magnitude low compared with the experimentally measured one. Thereby, $1/B$ anomalous diffusion (Bohm diffusion)\textsuperscript{2} is assumed in this calculation.

The magnetic fields are applied by a magnetic circuit consisting of solenoidal coils and magnetic pole pieces, and the induced magnetic fields by diamagnetic currents in plasma are neglected, since plasma pressure is much smaller than magnetic field pressure in the channel.

**Basic Equations**

Ions are accelerated by electrostatic fields and the equation of motions is expressed as

$$M \frac{dv}{dt} = eE$$

As for electrons, the diffusion current density is expressed by

$$J_e = en [\mu]E + e [D] \nabla n$$

Note that this equation is concerned only with the axial and radial electron motions, and not the azimuthal one. The electron mobility and the diffusion coefficient are expressed in tensors, because these coefficients are anisotropic in the presence of magnetic fields. The
coefficients parallel to magnetic field lines are given by the classical diffusion ones
\[ \mu_1 = \frac{e}{m_1v}, \quad D_1 = \frac{kT_e}{m_1v} \] (3)

The coefficients perpendicular to magnetic field lines are expressed by the Bohm diffusion ones
\[ \mu_\perp = \frac{1}{16\beta}, \quad D_\perp = \frac{kT_e}{16e\beta} \] (4)

In the acceleration channel, the mobility parallel to the field lines is four orders of magnitude larger than that perpendicular to the fields lines.

Ion and electron current conservation equations inside the channel are expressed by
\[ \nabla \cdot J_i = e\rho \] (5)
and
\[ \nabla \cdot J_e = -e\rho \] (6)

On the channel wall boundaries, the ions accelerated toward the channel walls are re-combined with electrons. When the channel wall is non-conducting, the ion current to the wall surface must be equal to the electron current. Thereby, the boundary condition of current density on the wall surface becomes
\[ (J_e)_n = (J_i)_n \] (7)
where \((J_e)_n\) and \((J_i)_n\) represent the electron and ion current density normal to the wall surface, respectively.

**Ion Production**

As for ion-production distribution, three cases are considered. In the first case, all of ions are assumed to be produced in a plasma source located at the upstream end of the acceleration channel. Ions supplied by the plasma source are assumed to be introduced into the channel with the Bohm velocity, which is the boundary condition necessary to obtain the solution of monotonic decrease in space potential in the axial direction. This case is also corresponding to the sheath type Hall thrusters in which ions are mostly produced in the thin sheath layer generated near the anode surface. On the other hand, in the case of single-stage discharge Hall thrusters, ion-production distribution spreads over the acceleration channel. Thereby, in the second case, several profiles of ion-production distribution are assumed, and the initial velocity of the produced ions is given as zero or very small.

In the third case, ion-production distribution is determined from the electron energy conservation equation in stead of being given as an input parameter for single-stage discharge Hall thrusters. The electron energy conservation equation can be expressed as
\[ \nabla \left( \frac{5}{2} kT_e \frac{J_e}{e} \right) = J_e E - (1+\gamma) e \nabla \cdot J_e \]
- \[ \nabla \cdot \left( \frac{5}{2} \nabla (kT_e) \right) \] (8)

This equation indicates that the electrons gain energy by Joule's heating and consume it in ionization and excitation in the channel. The last term on the right-hand side is associated with heat conduction. From the Reynolds analogy, heat conductivity is given as \([k] = 5/2[D]\). Substituting Eq. (6) into Eq. (8), ion-production rate appears explicitly in the equation as
\[ \left( \frac{5kT_e}{2e} + (1+\gamma)e \right) q = J_e E \]
\[ - \frac{5}{2} J_e \nabla \left( kT_e \right) \]
\[ + \nabla \left( \frac{5}{2} \nabla (kT_e) \right) \] (9)

Since electron temperature variation in the channel is usually much smaller than that of space potential (i.e. electric field \( E \)), the temperature gradient term \( \nabla (kT_e) \) is neglected here, and Eq. (9) reduces to
\[ q = \frac{J_e E}{\frac{5kT_e}{2e} + (1+\gamma)e} \] (10)

Here, electron temperature and excitation coefficient are assumed uniform in the channel and are given as input parameters on calculation. Although the absolute value of ion-production rate changes with these parameters, it is proportional to the Joule's heating so that the distribution's shape does not change. Consequently, these parameters have little influence on space potential distribution and on ion-loss fraction.

**Calculation Procedure**

There are two different parts of calculation; electron diffusion calculation and ion trajectory calculation. In order to apply the finite element method to the electron diffusion calculation, the region inside the channel is divided into small triangle elements, and these elements are also utilized for the ion trajectory calculation.

In the ion trajectory calculation, the ion flux produced in the k-th element is given by
\[ \Gamma_i^{(k)} = q^{(k)} V^{(k)} \] (11)
where \( q^{(k)} \) is the ion-production rate at the k-th element and \( V^{(k)} \) is its volume. When the space potential distribution
is known, one can trace the ion trajectory by integrating Eq. (1), which is transformed to the following equations.

\[ v_n = v_{n-1} + \frac{eE}{M} \Delta t \]  \hspace{1cm} (12)

\[ x_n = x_{n-1} + v_{n-1} \Delta t + \frac{eE}{2M} (\Delta t)^2 \]  \hspace{1cm} (13)

Here, \( v \) and \( x \) denote the ion velocity and position vector at the \( n \)-th iteration. Electric field is given as the gradient of space potential distribution

\[ E = -\nabla \phi \]  \hspace{1cm} (14)

Plasma density can be calculated by counting the number of ions passing through an element and their residence time in the element. The density at the \( l \)-th element \( n^{(l)} \) is given by

\[ n^{(l)} = \sum_{k=1}^{N} \frac{\Gamma^{(k)} l}{v^{(l)}} \]  \hspace{1cm} (15)

Here, \( \Gamma^{(k)} l \) represents the time of the respective ions in flux \( \Gamma^{(k)} \) to stay in the \( l \)-th element. The ion-loss current distribution on the wall surface is obtained by summing the ion flux arrived at the wall surfaces.

\[ J_{\text{wall}}^{(l)} = e \sum_{k=1}^{N} \frac{\Gamma^{(k)} l}{S^{(l)}} \]  \hspace{1cm} (16)

Here, \( J_{\text{wall}}^{(l)} \) is the ion current density normal to the \( l \)-th wall surface element and \( S^{(l)} \) is its surface area.

Space potential distribution is obtained by solving the electron diffusion equation, which is rewritten by combining Eqs. (2), (6) and (14).

\[ \nabla(n \mu \nabla \phi) = \nabla[D] \nabla n + q \]  \hspace{1cm} (17)

Here, the tensors of electron mobility and diffusion coefficient are expressed as

\[ [\mu] = [\Theta]^{-1} \begin{bmatrix} \mu_{\perp} & 0 \\ 0 & \mu_{/} \end{bmatrix} [\Theta] \]

\[ [D] = [\Theta]^{-1} \begin{bmatrix} D_{\perp} & 0 \\ 0 & D_{/} \end{bmatrix} [\Theta] \]  \hspace{1cm} (18)

at the point where the magnetic field line makes an angle of \( \theta \) relative to the axis. The rotation matrix \([\Theta]\) is defined as

\[ [\Theta] = \begin{bmatrix} \cos \theta \cdot \sin \theta \\ \sin \theta \cdot \cos \theta \end{bmatrix} \]  \hspace{1cm} (19)

The magnetic field is calculated prior to this analysis as a solution to the magnetostatic equations.

The boundary conditions of the space potential at the entrance and the exit of the channel are given as that it is equal to the anode potential \( \phi_{\text{anode}} \) at the entrance, and to zero at the exit.

\[ \phi_{\text{entrance}} = \phi_{\text{anode}} \quad \phi_{\text{exit}} = 0 \]  \hspace{1cm} (20)

The calculation sequence is illustrated in Fig. 2. At the beginning of the calculation, the channel geometry and magnetic field configuration are given as input parameters. The ion-production distribution is, as previously mentioned, assumed in the first and the second cases, and is self-consistently calculated in the third case. Assuming the initial space potential distribution, one can calculate the ion trajectories from Eqs. (12) and (13). The plasma density and ion-loss distributions are obtained from Eqs. (15) and (16), respectively. With these distribution profiles, the electron diffusion equation, Eq. (17) is solved on the boundary conditions given by Eqs. (7) and (20), and the space potential is renewed. In the third case, ion-production distribution is determined by

![Fig. 2 Calculation sequence of two-dimensional plasma flow analysis. (a) with assumed ion-production rate.](image-url)
solving Eq. (10) with the calculated space potential and plasma density distributions. These calculations are repeated by turns until both space potential and plasma density are converged within tolerable error. Finally, the ion-loss fraction is obtained by dividing the total ion-loss current on the wall surface by total ion-production current in the channel.

\[ \alpha = \frac{\int \mathbf{J}_i \cdot d\mathbf{S}}{\int q \, dV} \]  \hspace{1cm} (21)

**Results and Discussion**

For the first case, two patterns of magnetic field configurations are considered as shown in Fig. 3. In Pattern I, the magnetic field lines are curved and their direction changes from the downward to the upward in the channel, while in Pattern II, the magnetic field lines are formed almost straightly in the radial direction. As seen in the space potential profile in Pattern I, the equipotential lines are curved in accordance with the magnetic field lines. This is because electrons move freely along the magnetic field lines rather than across them and eliminate the potential difference along them. As a result, the radial component of electric field is induced. Owing to this field distortion, most of the ions introduced into the channel are accelerated toward the outer wall and the ion-loss fraction exceeded 0.9. On the
other hand, in the case of Pattern II, the electric fields are mostly formed in the axial direction, and the ion-loss fraction becomes as small as 0.1. Such a large reduction in ion-loss fraction is mainly due to the better arrangement of the magnetic field lines which are formed almost perpendicular to the axis. Consequently, the electric field distortion, which is the cause of ion-loss to the channel walls, is found to be induced by the curved configuration of the magnetic field lines.

For the second case, ion-production distribution is assumed to have a peak in the middle of the channel as shown in Fig. 4. Profiles of plasma properties are also shown in the figure. As seen in the space potential profile, though the axial electric fields are dominating in the channel, the radial component of the electric field is induced in the regions near the wall surfaces. This component is set up as a polarity as to retard the electron flux toward the wall, and is balanced with the density gradient which is made as to drive it, since the electron flux to the wall is limited as small due to the boundary condition (see Eqs. (2) and (7)). This phenomena is also observed in the experimental results.\(^7\) Owing to the radial electric field, the ions produced near the walls are accelerated toward the walls, and the ion-loss fraction becomes 0.25.

As the electron temperature increases, the electron diffusion coefficient becomes larger and the density gradient effect stronger, and consequently, the ion loss increases.

The radial density gradient, which might produce radial electric fields, is considered to come from the radial gradient of ion-production distribution. In order to examine the influence of the shape of ion-production distribution on the plasma density and space potential distributions, the case with radially uniform ion-production distribution was calculated. The results are shown in Fig. 5. In this case, the plasma density is distributed almost uniformly in the radial direction, and hence no radial electric field is induced. The computed ion-loss fraction becomes as small as 0.06. This result suggests that the ion-loss fraction can be reduced with the radially uniform ion-production distribution. In a practical case, it is, however, difficult to obtain such a uniform ion-production distribution, and thereby, there will be an optimum magnetic field configuration which cancels the effect of the ion-production gradient.

In the third case, calculation was conducted on two different Hall thrusters, and the results are compared with the measured ones. Thruster I is the one which was...
made at University of Tokyo and Thruster II is the one which was developed at the Moscow Aviation Institute in Russia. These thrusters had been experimentally investigated and the plasma properties' profiles in the channel had been measured with electrostatic probes. The computed results for Thruster I and for Thruster II are shown in Figs. 6 and 7, respectively. As seen in Fig 6, the ion-production distribution has a peak in the region very close to the outer wall. This is because the electron current density near the outer wall is relatively high (thereby, the production rate is high) due to the low magnetic induction and high electron mobility. Magnetic induction at the vicinity of the outer wall is about half of that near the inner wall in Thruster I. Half of the ions produced in this region were lost to the wall, resulting in a high ion-loss fraction of 0.55.

In the case of Thruster II, the calculated ion-loss fraction becomes 0.3. This value is about half of that of Thruster I. As seen in Fig. 7, the ion-production distribution is not biased toward the outer wall but has a peak in the center region of the channel. Furthermore, the peak of the ion-production rate shifts to the downstream region, where strong electric fields are induced. Owing to this distribution, ions are extracted from the channel exit with relatively small ion loss. Such magnetic field configuration (high magnetic induction near the exit) is a distinctive feature of the conventional Russian Hall thrusters. The computed profiles of ion-production distribution show good agreements with the measured ones.
Finally, the relationship between the ion-loss fraction and thruster performance is discussed. From a simple plasma discharge model (see Appendix), one can rewrite the acceleration efficiency (a ratio of ion-beam current to acceleration current) with using ion-loss fraction and ion-production coefficient as

$$\eta_a = \frac{1}{1 + 1/f(1 - \alpha)}$$  \hspace{1cm} (22)$$

Here, the ion-production coefficient is approximately derived from Eq. (10) as

$$\beta = \frac{V_a}{5kT^2/2e + (1+\gamma)e}$$  \hspace{1cm} (23)$$

assuming the electron current density is almost uniform. Equation (22) implies that high acceleration efficiency is obtained with low ion-loss fraction and with high ion-production coefficient. As previously shown in this paper, ion-loss fraction changed from 0.1 to 0.9 depending on the thruster configuration. On the other hand, ion-production coefficient is, as seen in Eq. (23), dependent on the operational parameters, and not on the geometrical parameters of thruster configuration.

The thrust efficiency is expressed as a product of three internal efficiencies (see Appendix) such as acceleration efficiency, propellant utilization and beam energy efficiency,

$$\eta_t = \eta_a \eta_u \eta_k$$  \hspace{1cm} (24)$$

Among these internal efficiencies, the acceleration efficiency is usually the lowest one and is a key parameter for performance improvement. Therefore, the improvement of the acceleration efficiency by reducing the ion-loss fraction, is the most effective way to achieve high thrust efficiency.

**Summary**

A two-dimensional plasma flow model, which is comprised of an electron diffusion equation and an ion kinetic equation, has been made to compute ion-loss flux to the surrounding walls of an acceleration channel.

In the first step, the distributions of plasma properties such as space potential, plasma density and ion-loss rate are calculated, and the ion-loss fraction is estimated. As a result, it was found that the radial electric field, which is a main cause of ion-loss to the walls, is induced due to the radial gradient of ion-production distribution in addition to the curvature of the magnetic field lines. Therefore, channel geometry and magnetic field configuration should be determined after taking account of the effect of ion-production distribution on ion-loss flux to the walls.

In the second step, ion-production distribution was self-consistently determined from the energy conservation equation coupled with the above equations. The result indicates that the ion-production distribution changes largely with the magnetic induction distribution, and hence the magnetic field configuration is found to have a great influence on the ion-loss fraction and on the thruster performance. The calculated results were compared with the measured ones, and have shown good agreements. From this fact, it was concluded that this model would be a useful tool for estimation of ion-loss fraction in the acceleration channel.

In order to predict the thruster performance more precisely, accurate production-rate estimation is required. Therefore, future work should focus on determining the electron temperature, which is assumed constant in this model. For this purpose, the equations concerning on the ionization process and on the neutral gas flow should be solved.

**Appendix : Definition of the Efficiencies**

From thrust measurement, one can calculate thrust efficiency using the well-known relation as given by

$$\eta_t = \left( \frac{F^2}{2mV_a I_a} \right)$$  \hspace{1cm} (A1)$$

Neither excitation power for solenoidal coils nor heating power for cathode neutralizer is taken into account on the calculation of thrust efficiency, since they are much smaller than that for main discharge.

In order to investigate thruster performance characteristics, following internal efficiencies are introduced and defined by the equations:

$$\eta_a = \left( \frac{I_b}{I_a} \right)$$  \hspace{1cm} (A2)$$

$$\eta_u = \left( \frac{Ml_b}{em} \right)$$  \hspace{1cm} (A3)$$

$$\eta_E = \left( \frac{V_m}{V_a} \right)$$  \hspace{1cm} (A4)$$

Here, $V_a$ is the mean beam-ion energy which is calculated from the ion-energy distribution as

$$V_m = \left( \int f(V) \sqrt{2V} dV \right)$$  \hspace{1cm} (A5)$$

When all ions are singly-charged and are accelerated only in the axial direction, thrust can be written by...
Substituting Eqs. (A2-4) and (A6) into Eq. (A1), the thrust efficiency yields

$$\eta = \eta_0 \eta_1 \eta_2$$

(A7)

Typical examples of such internal efficiencies and thrust efficiency for several operating conditions are listed in Table A1. As seen in this table, the acceleration efficiency is the most predominant factor for determining thrust efficiency.

To evaluate the acceleration efficiency in detail, a simple model is presented as follows. In the acceleration channel, ions are produced by ionization collisions of electrons with neutral atoms, and total ion-production current in the channel can be expressed by

$$I_p = \beta I_e$$

(A8)

where \(I_e\) is electron current backstreaming from the cathode to the anode and a coefficient \(\beta\) is a quantity that expresses how efficiently ions are produced. As the volume recombination in the channel can be neglected, the ions produced are either lost to the channel walls or exhausted downstream from the exit. Thereby, \(I_p\) is given by

$$I_p = (1 - \alpha) I_e$$

(A9)

where, \(\alpha\) denotes a fraction of ions produced that are lost to the walls.

The sum of the ion-beam current and the electron current is equal to the acceleration current as

$$I_a = I_b + I_e$$

(A10)

Substituting Eqs. (A8-10) into Eq. (A2), the acceleration efficiency can be expressed as a function of \(a\) and \(b\) as

$$\eta_a = \frac{1}{1 + 1/\beta (1 - \alpha)}$$

(A11)

Table A1 Typical internal efficiencies

<table>
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<th>(m(A\text{-eq}))</th>
<th>(V_e(V))</th>
<th>(I_a(A))</th>
<th>(\eta_0(%))</th>
<th>(\eta_1(%))</th>
<th>(\eta_2(%))</th>
<th>(\eta_i(%))</th>
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<td>13</td>
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<tr>
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<td>29</td>
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Propellant gas is xenon.