A Stationary Hydrodynamic Model for Hall Thruster*†

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A new stationary hydrodynamic model of the Hall thruster has been developed. The analysis of this model shows that for typical SPT operating conditions, a relatively thin layer with large gradients of plasma parameters can exist within the flow inside the SPT channel. A considerable part of the ion acceleration occurs within this layer, whose thickness can be determined by the electron viscosity. In good agreement with the predictions of our model, recent LIF measurements show a clear manifestation of ion acceleration in a relatively thin layer in the near-cathode region of the thruster.

1. Introduction

For the first time, the stationary plasma thrusters (SPT) or Hall thrusters were suggested and developed in the Soviet Union by A. I. Morozov [1]. By now, many of the Russian spacecrafts were equipped by the SPT. These thrusters are widely used in the active space experiments as well.

Although a good deal of investigation is already done in this field (see reviews [1–5]), the physical processes within the thruster channel are far from being completely understood. For example, many questions concerned with not only oscillatory regimes but with the quasistationary nominal regime with the low level of oscillations still remain to be answered. To study the nominal regime of SPT operation, several relatively simple hydrodynamic models were suggested (see, e.g., [6–8]).

In the present paper we develop a new stationary hydrodynamic model for Hall thrusters. Our model is more general than the models, which were suggested previously.

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Table 1. Typical parameters of plasma flow within the SPT channel

	Near-anode region	Acceleration zone
Electron temperature, Te (eV)	5.5	14.8
Number density of neutrals, cm^{-3}	$9.5\cdot10^{14}$	$1.2\cdot10^{13}$
Number density of electrons, cm^{-3}	$9.7\cdot10^{10}$	$7.1\cdot10^{11}$
Axial ion velocity, cm/s	0.13	$8.2\cdot10^{4}$

The paper is organized as follows. In the 2nd section we discuss the typical values of different plasma parameters within the SPT channel. Some of these parameters are directly measured, the others can only be estimated. In the 3rd section the basic equations governing the dynamics of plasma within the channel are presented. These equations are analyzed in detail in the 4th section. The main results of the analysis are summarized in the 5th section of the paper.

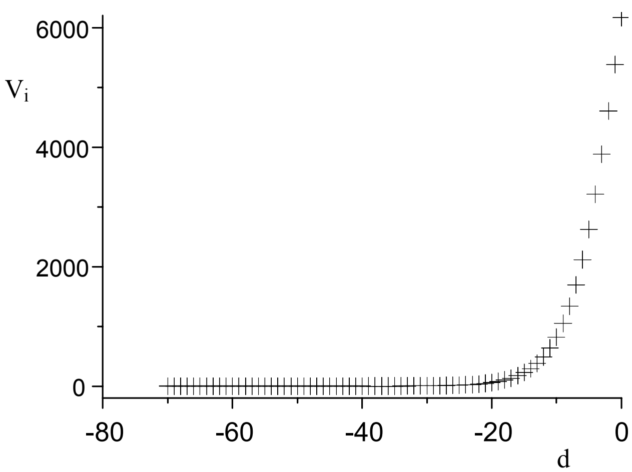

Figure 1. Ion axial velocity V_i (m/s) versus distance d (mm) from the exit plane.

2. Typical Plasma Parameters Within the SPT Channel

To investigate the physical processes within the SPT channel, the reliable values of different parameters are of crucial importance. The most of these parameters have already been measured, but the experiments were carried out with different SPT models and/or for different regimes of SPT operation. Now for most of SPT models we have no comprehensive data sets.

In our study we consider a low-power Hall thruster, which is developed at Stanford University and described elsewhere [9, 10].

We split the thruster channel into two parts, namely, near-anode region and acceleration zone, which is located in the vicinity of the exit. The plasma parameters, which can be considered as typical for these two zones within the thruster channel, are presented in Table 1. For definiteness, we take data obtained for discharge voltage 160 V [11]. Figure 1 shows the laser induced fluorescence (LIF) measurements of axial ion velocity [11]. The total length of the thruster channel is 80 mm. The distance is measured from the exit plane, the negative coordinates correponding to the interior of the thruster.

3. Hydrodynamic Equations Describing Stationary Regimes of the SPT

3.1. The Basic Equations

Strictly speaking, the behavior of all plasma components within the SPT channel can be described only by the corresponding kinetic equations in the six-dimensional phase space for each component and we should take into account both the elastic and inelastic collisions of the particles considered with each other and with the walls of the channel. Indeed, the experiments show that the profiles of different physical quantities describing the plasma flow are essentially two-dimensional (see, e.g., [12]). The electron distribution function is quite complicated and can be approximately considered as having three components [13]. Simple estimates show that the most of ions and neutrals suffice only a few collisions when they transit the channel. However, obtaining the solution of three coupled kinetic equations is a very complicated problem requiring extensive numerical computations. Furthermore, one can encounter with significant difficulties when interpreting the results obtained. Therefore, to understand the physics of the processes within the thruster channel, we need a simple mathematical model, which can easily be investigated analytically or numerically. In this section we describe a one-dimensional stationary model based on hydrodynamical description of plasma components.

Nevertheless, we write the basic equations in the time-dependent vector form and the simplifications mentioned above will be made later.

3.1.1. Neutrals

Suppose that all the neutrals leave the anode with the same velocity $\mathbf{V}_n$, directed along the thruster axis. If we neglect the recombination and charge exchange processes, this velocity does not vary,

$$\mathbf{V}_n = \text{const.}$$

In addition, we have the continuity equation for neutrals,

$$\frac{\partial N_n}{\partial t} + \operatorname{div}(N_n \mathbf{V}_n) = -K_{\text{ion}} N_n N_e,$$

where K_{ion} is the ionization rate constant.

3.1.2. Ions

The continuity equation for ions is given by

$$\frac{\partial N_e}{\partial t} + \operatorname{div}(N_{\text{e}} \mathbf{V}_i) = K_{\text{ion}} N_n N_e.$$

Here and in the following we use the quasineutrality assumption, i.e., we suppose that the number densities for electron and ions are approximately equal and introduce for them the same notation N_e.

To obtain the hydrodynamic equation of motion for ion component, we can choose a small stationary volume and consider the balance of momentum of ions within this volume. Thereby we get

$$\frac{\partial}{\partial t}(MN_e V_i) + \frac{\partial}{\partial x_k}(MN_e V_i V_k) = eN_e E_i + MV_{ni} K_{\text{ion}} N_n N_e,$$

where V_{ni} is the ith component of the velocity of neutrals and summation over the repeating indices is assumed. The last term describes the change of momentum due to ionization processes. Combining this equation with the continuity equation, we obtain the equation of motion for ion component,

$$\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_{\text{i}} \cdot \nabla)\, \mathbf{V}_{\text{i}} = \frac{e}{M} \mathbf{E} + (\mathbf{V}_{\text{n}} - \mathbf{V}_{\text{i}})\, K_{\text{ion}} N_n N_e.$$

We observe that the ionization results in the effective "retardation" of ions.

3.1.3. Electrons

For electrons we have the continuity equations, similar to that of ions,

$$\frac{\partial N_e}{\partial t} + \operatorname{div}(N_{\text{e}} \mathbf{V}_e) = K_{\text{ion}} N_n N_e,$$

the equations of motion [14],

$$mN_e \frac{d_e V_{ep}}{dt} = -\frac{\partial P_e}{\partial x_p} - \frac{\partial \pi_{pq}}{\partial x_q} - eN_e \left(E_p + \frac{1}{c} \varepsilon_{pqr} V_{eq} B_r \right) + R_p,$$

and the energy balance equation,

$$\frac{\partial}{\partial t}\left(\frac{N_e m \mathbf{V}_{\mathbf{e}}^2}{2} + N_e I + \frac{3}{2} N_e T_e \right) + \frac{\partial}{\partial x_p}\left[V_{ep}\left(\frac{N_e m \mathbf{V}_e^2}{2} + N_e I + \frac{5}{2} N_e T_e \right) + q_p + \pi_{pq} V_{eq} \right] = -eN_e \mathbf{E}\mathbf{V}_e + \mathbf{R}\mathbf{V}_e + Q + Q_{\text{wall}}.$$

Here ε_{pqr} is the antisymmetric unit tensor and

$$\frac{d_e}{dt} = \frac{\partial}{\partial t} + V_{ep} \frac{\partial}{\partial x_p}.$$

In contrast to the energy balance equation in [14], we include the terms due to ionization/recombination processes and interaction of the plasma with walls of the channel. In these equations, $P_e = N_e T_e$ is the electron pressure, T_e is the electron temperature, I is the ionization threshold, π_{pq} is a stress tensor, by Q_{wall} we denote the heat transferred from the walls of the channel to electron gas, $\mathbf{R} = \mathbf{R}_{\text{en}} + \mathbf{R}_{\text{ei}}$ is the change of momentum due to collisions of electrons with neutrals and ions, $Q = Q_{\text{en}} + Q_{\text{ei}}$ is the corresponding heat produced in the electron gas. We have [14]

$$Q_{ei} = \mathbf{R}_{ei}(\mathbf{V}_i - \mathbf{V}_e), \quad Q_{en} = \mathbf{R}_{en}(\mathbf{V}_n - \mathbf{V}_e).$$

Since, as a rule, the velocity of electrons is much greater than that of neutrals and ions, we get

$$Q \approx -\mathbf{R}\mathbf{V}_e.$$

The force exerting on the electron gas can be divided into two parts, i.e., a friction force $\mathbf{R}_u$ and a thermal force $\mathbf{R}_T$,

$$\mathbf{R}_e = \mathbf{R}_u + \mathbf{R}_T,$$

These forces are given by [14]

$$\mathbf{R}_u = -\alpha_{\parallel} \mathbf{u}_{\parallel} - \alpha_{\perp} \mathbf{u}_{\perp} + \alpha_{\wedge} [\mathbf{h}, \mathbf{u}],$$

$$\mathbf{R}_T = -\beta_{\parallel}^{ut} \nabla_{\parallel} T_e - \beta_{\perp}^{uT} \nabla_{\perp} T_e - \beta_{\wedge}^{uT} [\mathbf{h}, \nabla T_e],$$

where $\mathbf{u} = \mathbf{V}_e - \mathbf{V}_{\text{n,i}} \approx \mathbf{V}_e$. Analogously, the electron heat flux is [14]

$$\mathbf{q}_e = \mathbf{q}_u + \mathbf{q}_T,$$

where

$$\mathbf{q}_u = \beta_{\parallel}^{Tu}\mathbf{u}_{\parallel} + \beta_{\perp}^{Tu}\mathbf{u}_{\perp} + \beta_{\wedge}^{Tu}[\mathbf{h},\mathbf{u}]$$

and

$$\mathbf{q}_T = -\kappa_{\parallel}^{e}\nabla_{\parallel}T_e - \kappa_{\perp}^{e}\nabla_{\perp}T_e - \kappa_{\wedge}^{e}[\mathbf{h},\nabla T_e].$$

The coefficients $\alpha_{\parallel}$, $\beta_{\parallel}^{Tu}$, and $\kappa_{\parallel}^{e}$ will not be used in the following. If the electrons are magnetized, i.e., $\delta_e = 1/\omega_e\tau_e << 1$ (ω_e is the electron gyrofrequency and τ_e is a characteristic time interval between the collisions distorting the electron trajectory), the other coefficients are given by

$$\alpha_{\perp} = \frac{mN_e}{\tau_e}, \quad \alpha_{\wedge} = \frac{mN_e}{\tau_e}\alpha_1''\delta_e,$$

$$\beta_{\perp}^{uT} = N_e\beta_1'\delta_e^2, \quad \beta_{\wedge}^{uT} = N_e\beta_1''\delta_e,$$

$$\beta_{\perp}^{Tu} = T_e\beta_{\perp}^{uT}, \quad \beta_{\wedge}^{Tu} = T_e\beta_{\wedge}^{uT};$$

$$\kappa_{\perp}^{e} = \frac{N_eT_e\tau_e}{m}\gamma_1'\delta_e^2, \quad \kappa_{\wedge}^{e} = \frac{N_eT_e\tau_e}{m}\gamma_1''\delta_e,$$

where for coefficients we use the notation suggested in [14]. If the Coulomb collisions dominate, we can use the values given in Table 2 (see, e.g., [14]).

Table 2. The coefficients for calculating transport properties of a plasma with Coulomb collisions

α_1''	β_1'	β_1''	γ_1'	γ_1''	ς_0	ς_2	ς_4
1.7	5.1	3/2	4.7	5/2	0.7	2.1	1.0

The stress tensor π_{pq} is given by

$$\pi_{\mathrm{pq}} = -\eta_0 W_{(0)pq} - \eta_1 W_{(1)pq} - \eta_2 W_{(2)pq} + \eta_3 W_{(3)pq} + \eta_4 W_{(4)pq},$$

where the quantities $W_{(i)pq}$, i = 0, ... , 4, can be expressed in terms of the rate-of-strain tensor,

$$W_{pq} = \frac{\partial V_{ep}}{\partial x_q} + \frac{\partial V_{eq}}{\partial x_p} - \frac{2}{3}\delta_{pq}\frac{\partial V_{er}}{\partial x_r},$$

where δ_{pq} is the Kronecker delta. If the magnetic field is directed along the z-axis, we have

$$W_{(0)xx} = W_{(0)yy} = 1/2(W_{xx} + W_{yy}), \quad W_{(0)zz} = W_{zz},$$

$$W_{(1)xx} = -W_{(1)yy} = 1/2(W_{xx} - W_{yy}),$$

$$W_{(1)xy} = W_{(1)yx} = W_{xy}, \quad W_{(2)xz} = W_{(2)zx} = W_{xz},$$

$$W_{(2)yz} = W_{(2)zy} = W_{yz}, \quad W_{(3)xx} = -W_{(3)yy} = -W_{xy},$$

$$W_{(3)xy} = W_{(3)yx} = 1/2(W_{xx} - W_{yy}),$$

$$W_{(4)xz} = W_{(4)zx} = -W_{yz}, \quad W_{(4)yz} = W_{(4)zy} = W_{xz},$$

the other components are equal to zero. If, again, $\delta_e = 1/\omega_e\tau_e << 1$, the viscosity coefficients are given by

$$\eta_0 = \varsigma_0 N_e T_e \tau_e,$$

$$\eta_2 = \varsigma_2 N_e T_e \delta_e^2, \quad \eta_1 = \eta_2(\delta_e/2),$$

$$\eta_2 = -\varsigma_4 N_e T_e \tau_e \delta_e, \quad \eta_3 = \eta_4(\delta_e/2),$$

where the coefficients for Coulomb collisions are given in Table 2 (see, e.g., [14]).

3.2. The Model Equations

We analyze the thruster model in the Cartesian coordinates. Let x-axis is directed along the symmetry axis of the thruster, the origin being at the anode. The thruster channel is considered as a flat slab, which is perpendicular to z-axis and infinite in the y-direction corresponding to azimuthal direction in the real thruster. The magnetic field is parallel to z-axis corresponding to radial direction in the thruster. Suppose that all parameters describing the plasma flow can change only in the x-direction and does not depend on time, i.e., we seek a stationary one-dimensional solution. The velocity of neutrals is along the x-axis.

Combining the continuity equations for all plasma species, we easily obtain

$$N_e(V_i - V_{ex}) = j/e,$$

where j is the total current density,

$$N_n V_n + N_i V_i = m/M.$$

These equations allow us to exclude the number density of neutrals and the ion velocity along the thruster channel. Then from the electron continuity equation we obtain

$$\frac{d(N_e V_{ex})}{dx} = K_{\mathrm{ion}} N_n N_e.$$

The x-components of the equations of motion for ions and electrons are as follows:

$$MN_eV_i\frac{dV_i}{dx} = eN_eE + K_{\mathrm{ion}}N_nN_e(V_n - V_i), \quad (1)$$

$$mN_eV_{ex}\frac{dV_{ex}}{dx} = eN_e\left(E - \frac{1}{c}V_{ey}(B + B')\right) - \frac{dP_e}{dx} - \frac{d\pi_{xx}}{dx} + R_x, \quad (2)$$

where B is an external magnetic field and B' is a disturbance of the magnetic field due to plasma currents in the channel.

If we consider only large-scale plasma motions such that the characteristic length is much greater than the Debye length, then the quasineutrality holds, i.e., the difference between the number densities of electrons and ions is much less than the electron density,

$$|N_e - N_i| << N_e.$$

In this case the Poisson equation is not required, instead, the electric field can be determined from the equations of motion for electrons at the final stage of analysis upon obtaining the profiles for all hydrodynamic quantities. At the first stage, the electric field can be considered as an auxiliary quantity and it is convenient to exclude it. Subtracting Eq. (1) from (2), we get

$$\begin{aligned} MN_e V_i \frac{dV_i}{dx} - mN_e V_{ex}\frac{dV_{ex}}{dx} \\ = K_{\text{ion}} N_n N_e (V_n - V_i) \\ + \frac{e}{c} N_e V_{ey}(B + B') + \frac{dP_e}{dx} + \frac{d\pi_{xx}}{dx} - R_x. \end{aligned} \tag{3}$$

In addition, we have the y-component of the electron equation of motion,

$$V_{ex}\frac{dV_{ey}}{dx} = -\frac{d\pi_{xy}}{dx} - \frac{1}{c} N_e V_{ex} B_z + R_y, \tag{4}$$

a stationary energy balance equation,

$$\begin{aligned} \frac{\partial}{\partial x}\left[V_{ex}\left(\frac{N_e m \mathbf{V}_e^2}{2} + N_e I + \frac{5}{2} N_e T_e \right) \right. \\ \left. + q_x + \pi_{xx} V_{ex} + \pi_{xy} V_{ey} \right] \\ = -eN_e E V_{ex} + \mathbf{R}\mathbf{V}_e + Q + Q_{\text{wall}}, \end{aligned} \tag{5}$$

and a Maxwell equation,

$$\frac{dB'}{dx} = -\frac{4\pi e}{c} N_e V_{ey}. \tag{6}$$

To proceed further, we must substitute the expressions for the friction force, heat flux, stress tensor, and heat production considered in the previous subsection into Eqs. (3)–(6). For simplicity, we neglect the energy exchange between electrons and walls of the channel, $Q_{\text{wall}} \approx 0$.

To analyze the system of equations obtained, it is useful to rewrite it in terms of the dimensionless variables,

$$\widetilde{N}_n = N_n / N_{n0}, \quad \widetilde{V}_{ex} = V_{ex} / V_{ex0},$$

$$\widetilde{V}_{ey} = V_{ey} / V_{ey0}, \quad \widetilde{T} = T / T_0,$$

where the values N_{n0}, V_{ex0}, V_{ey0}, and T_0 used for normalization are arbitrary and can be chosen from the consideration of convenience.

Thereby we obtain the following system of equations (for brevity, we omit the tildes over the dimensionless variables and the subscript e for electron velocity):

$$N_n = C - DNV_x, \tag{7}$$

$$\frac{d}{dx}(NV_x) = \frac{N_n N}{\Delta_{\text{ion}}}, \tag{8}$$

$$\begin{aligned} \mu NV_x \frac{dV_x}{dx} + (NV_x + A)\left(\frac{dV_x}{dx} - \frac{A}{N^2}\frac{dN}{dx} \right) \\ - \frac{N_n N}{\Delta_{\text{ion}}}\left(\xi - V_x - \frac{A}{N} \right) + \frac{1}{\mathbf{M}^2}\frac{d}{dx}(NT) + \frac{\mu}{\gamma\delta l} NV_y \\ + \frac{\mu}{l} NV_x + \frac{d}{dx}\left[-\Delta_{vxx}\frac{dV_x}{dx} + \Delta_{vxy}\frac{dV_y}{dx} \right] = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\mu}{\gamma\delta} NV_x \frac{dV_y}{dx} = \frac{\mu}{\delta^2 l} NV_x - \frac{\mu}{\gamma\delta l} NV_y \\ - \frac{\beta_1''}{\mathbf{M}^2} N \frac{dT}{dx} + \frac{d}{dx}\left[\widetilde{\Delta}_{vxx}\frac{dV_x}{dx} + \widetilde{\Delta}_{vxy}\frac{dV_y}{dx} \right], \end{aligned} \tag{10}$$

$$\begin{aligned} N\left(\frac{3}{2}V_x - \frac{2\beta_1''\delta}{\gamma} V_y \right)\frac{dT}{dx} + NT\frac{dV_x}{dx} \\ - \frac{\beta_1'' T}{\gamma}\frac{d}{dx}(\delta N V_y) = -\frac{N_n N}{\Delta_{\text{ion}}} \\ \times\left[\varepsilon + \frac{3}{2}T + \frac{\mu \mathbf{M}^2}{2\gamma^2}\left(\gamma^2 V_x{}^2 + V_y{}^2 \right) \right] \\ + \frac{1}{\Delta_J}\left(\gamma^2 V_x{}^2 + V_y{}^2 \right) + \frac{d}{dx}\left(\Delta_T \frac{dT}{dx} \right) \\ + \mathbf{M}^2 \Delta_{vxx}\left(\frac{dV_x}{dx} \right)^2 + \frac{\delta}{\gamma}\mathbf{M}^2 \widetilde{\Delta}_{vxy}\left(\frac{dV_y}{dx} \right)^2, \end{aligned} \tag{11}$$

$$\frac{dB'}{dx} = -\frac{NV_y}{L}. \qquad (12)$$

Here we introduce a set of characteristic lengths,

$$\Delta_{\text{ion}} = \frac{V_{ex0}}{K_{\text{ion}} N_{n0}},$$

$$l = V_{ex0}\tau, \quad L = \frac{\gamma}{\delta l}\left(\frac{c}{\omega_{pe}}\right)^2,$$

$$\Delta_{vxx} = \frac{0.24\, l}{\mathbf{M}^2} NT, \quad \Delta_{vxy} = \frac{\delta l}{2\gamma \mathbf{M}^2} NT,$$

$$\tilde{\Delta}_{vxx} = \frac{l}{2\mathbf{M}^2} NT, \quad \tilde{\Delta}_{vxy} = \frac{0.51\delta l}{\gamma \mathbf{M}^2} NT,$$

$$\Delta_T = \frac{\gamma_1' \delta^2 l}{\mu \mathbf{M}^2} NT, \quad \Delta_J = \frac{\gamma^2 l}{\mu \mathbf{M}^2} \frac{1}{N},$$

and a set of dimensionless constants,

$$A = -\frac{I_d}{I_{e0}}, \quad C = -\frac{m/M - j/e}{N_{n0} V_n}, \quad D = \frac{N_{e0} V_{ex0}}{N_{n0} V_n},$$

$$\mu = \frac{m}{M}, \quad \xi = \frac{V_n}{V_{ex0}}, \quad \gamma = \frac{V_{ey0}}{V_{ex0}}.$$

By $\mathbf{M}$ we denote an electron "Mach number," $\mathbf{M}^2 = MV_{ex0}^2 / T_0$.

The characteristic lengths, which are proportional to l, depend on the electron collision frequency $1/\tau$, which is a sum of frequencies for Coulomb collisions and elastic collisions with neutrals (we do not take into account effective collisions of electrons due to plasma turbulence because there exist no reliable estimates of the corresponding frequency),

$$\frac{1}{\tau} = \nu_c + \nu_n,$$

$$\nu_c = \frac{4(2\pi)^{1/2} \Lambda e^4 N_e}{3m^{1/2} T^{3/2}}, \quad \nu_n = N\delta_t V_{Te},$$

where σ_t is the momentum transfer cross section, σ_t = $(0.9\text{–}2.5)\cdot 10^{-15}$ cm^2 [15], Λ is the Coulomb logarithm, which for low electron temperatures, $T_e < 50$ eV, is given by [14]

$$\Lambda = 23.4 - 1.15 \log N_e + 3.45 \log T_e$$

(here the number density and temperature of electrons are measured in cm^{-3} and eV, respectively).

These characteristic frequencies calculated for typical plasma parameters within the thruster channel (see Table 1) are shown in Table 3.

Table 3. Typical frequencies for elastic collisions of electrons in the SPT channel

	near-anode region	acceleration zone
ν_c, s^{-1}	$2.8\cdot 10^5$	$4.8\cdot 10^5$
ν_n, s^{-1}	$1.3\cdot 10^8$	$2.8\cdot 10^6$

We observe that the elastic collisions with netrals dominate in the regions considered.

Before proceeding to analysis of the system in detail, we note that there exists a characteristic length

$$L_B \cong (0.5 \div 1) \text{ cm}$$

determined by the profile of the external magnetic field. The other important scale, which does not appear in the equations, but determine the conditions for these equations to be applicable, is the electron Larmor radius,

$$\rho = \frac{V_{Te}}{\omega_e},$$

where V_{Te} is the electron thermal velocity, $V_{Te} = (T_e / m)^{1/2}$. The following relationship holds:

$$\frac{\rho}{l} = \frac{\delta}{\mu^{1/2} \mathbf{M}}.$$

The values of the characteristic lengths may differ by several orders of magnitude. This fact is illustrated by Table 4, where shown are the values of these scales in the near-anode region and acceleration zone.

Table 4. Typical values of some characteristic lengths for plasma flow in the SPT channel

	near-anode region	acceleration zone
Δ_{ion}, cm	0.43	0.33
l, cm	$3.2\cdot 10^{-2}$	$1.5\cdot 10^{-1}$
Δ_{vxx}, cm	$3.5\cdot 10^{-5}$	$3.2\cdot 10^{-2}$
Δ_{T}, cm	5.0	$7.9\cdot 10^{-2}$

The analysis of these equations, numerical and/or analytical, is a rather complicated problem. The first stage of the analysis is usually carried out with the use of asymptotic method of multiple scales (see, e.g., [16]). This method was proved to be efficient in the analysis of fine structure of shocks in gases and plasmas (see, e.g., [17]). In addition, it allows one to determine the physical mechanisms responsible for the formation of different structures in the gas or plasma flow. In subsection 4.1, using a relatively simple example concerned with the plasma flow in the near-anode region, we dwell upon some details of the asymptotic analysis procedure.

Finally, we point out the obvious fact that the largest scale for the plasma flow within the thruster channel cannot exceed the value L_B, which can be significantly less than the thruster channel length or has the same order of magnitude.

4. Analysis of the Model Equations

When analyzing a plasma flow in some region of the thruster channel, it is convenient to choose some point within this region and use the values of N_0, V_{x0}, V_{y0}, and T_0 at this point for normalization. Then all the values of N, $V_{x,y}$ and T are of the order of unity within the region considered. The constant γ remains arbitrary and must be determined during analysis from physical considerations.

4.1. Near-Anode Region

Table 4 shows that the characteristic length Δ_T is the largest in the near-anode region and

$$\Delta_{\text{ion}} \le L_B << \Delta_T.$$

Because Δ_T is too large, we try to find a solution corresponding to the largest of the remaining scales, i.e., L_B and Δ_{ion}. At first, we suppose that the change for all the quantities on the scale considered is of the order of unity. Next, in each equation we find the leading terms having the largest magnitude.

Equations (7) and (8) require no transformation. In Eq. (9) we observe that the terms due to electron inertia and viscosity are negligible. Indeed, the first term in Eq. (9) is much less than the second. The left-hand side in Eq. (10) become comparable with the 2nd term on the right-hand-side when the characteristic scale of the plasma flow is about l, on larger scales the left-hand-side is small, and so on. Omitting these small terms, from Eq.(10) after some straightforward manipulations we obtain

$$V_y = \frac{\gamma}{\delta} V_x - \frac{\beta_1'' \gamma \delta l}{\mu \mathbf{M}^2} \frac{dT}{dx}. \tag{13}$$

We can use this relationship for eliminating V_y from the remaining equations.

Substituting Eq. (13) into Eq. (10), combining it with the continuity equation (8), and neglecting some small terms, we get

$$\begin{aligned} &\left[\left(1+\frac{A}{NV_x}\right)^2 - \frac{T}{\mathbf{M}^2 V_x^{\,2}}\right]\frac{dV_x}{dx} \\ &= \frac{N_n}{\Delta_{\text{ion}}}\left[\left(\frac{A}{NV_x}\right)^2 - 1 + \frac{\xi}{V_x} - \frac{T}{\mathbf{M}^2 V_x^{\,2}}\right] \\ &- \frac{1}{\mathbf{M}^2 \Delta_0} + \frac{\beta_1''-1}{\mathbf{M}^2 V_x}\frac{dT}{dx}, \end{aligned} \tag{14}$$

where we introduced a characteristic length

$$\Delta_0 = \frac{\delta^2 l}{\mu \mathbf{M}^2},$$

which is comparable with Δ_T.

From the energy balance equation and we obtain

$$\begin{aligned} &(\gamma_1'' - \beta_1'') T \frac{d^2 T}{dx^2} + \left(\gamma_1'' - \beta_1''^2\right)\left(\frac{dT}{dx}\right)^2 \\ &+ (\gamma_1' - \beta_1'') T \frac{dT}{dx}\frac{d}{dx}\log\left(\frac{N}{B^2}\right) \\ &= -\frac{V_x^{\,2}}{\Delta_0^{\,2}} + \frac{1}{\Delta_0}\left\{\frac{3}{2}V_x\frac{dT}{dx} + T\frac{dV_x}{dx}\right. \\ &\left. + \frac{N_n}{\Delta_{\text{ion}}}\left[\varepsilon + \left(\frac{3}{2} - \beta_1'' T\right)\right]\right\} \end{aligned} \tag{15}$$

(for simplicity, we suppose that τ does not vary significantly on the scales considered). Since $\Delta_0 >> \Delta_{\text{ion}}$, the right-hand-side in (15) is small and we can neglect it. Upon integrating the equation obtained, we get

$$\frac{dT}{dx} = \text{const}\,\frac{B^2}{N} T^{(\gamma_1' - \beta_1'')/(\gamma_1' - \beta_1''^2)}, \tag{16}$$

where the constant can be determined from the boundary conditions. From Eq. (16) it is easily seen that there exist the solution

$$T = \text{const},$$

although the other quantities does change.

Note that the factor before the derivative of the electron velocity [see Eq. (14)] may vanish and as a result, a singularity may occur in the solution. This factor vanishes if $V_i^2 = T / M$ (here the variables are not dimensionless). Numerical integration of the system of equations (7), (8), (14), (16) with different values of the temperature gradient at the initial point shows that solutions with singularity do exist. When approaching such a singularity, the gradients are increased and the neglected terms become significant. When we take these terms into account, the discontinuity disappears, instead we obtain a thin layer with large but finite gradients. This thin layer is embedded into the plasma flow with small gradients. The largest of the remaining scales are related to electron viscosity. Thus, it is natural to suppose that this "discontinuity", which in the following will be called "an acceleration zone," is formed by electron viscosity.

4.2. Acceleration Zone

In the point considered to be typical for acceleration zone, we have

$$\Delta_{\text{ion}} << \Delta_T << \Delta_{vxx}.$$

Δ_{ion} is again the largest but it is small enough thereby demonstrating that the ionization processes are also significant in this region. Proceeding to smaller scales, Δ_{vxx} , we observe that the terms due to electron thermal conductivity are dominant in the energy balance equation (11). From this equation it follows that the temperature variations on such scales can be neglected. As in the case of larger scales, we omit the electron inertia terms. In addition, the Lorentz force is small as compared to the forces due to viscosity. Neglecting the small terms, from Eqs.(8)–(10) we obtain

$$\frac{d}{dx}\left(NV_x\right) = 0, \tag{17}$$

$$\begin{aligned} (NV_x + A)\left(\frac{dV_x}{dx} - \frac{A}{N^2}\frac{dN}{dx}\right) + \frac{1}{\mathbf{M}^2}\frac{d}{dx}(NT) \\ + \frac{d}{dx}\left[-\Delta_{vxx}\frac{dV_x}{dx} + \Delta_{vxy}\frac{dV_y}{dx}\right] = 0, \end{aligned} \tag{18}$$

$$\frac{d}{dx}\left[\widetilde{\Delta}_{vxx}\frac{dV_x}{dx} + \widetilde{\Delta}_{vxy}\frac{dV_y}{dx}\right] = 0. \tag{19}$$

Integrating Eq. (19), we get

$$\frac{dV_y}{dx} = -\frac{\gamma}{\delta}\frac{dV_x}{dx}. \tag{20}$$

The integration constant is equal to zero in the first approximation, because all the derivatives should vanish at the boundaries of the thin layer within the acceleration zone. Combining Eq. (20) with Eq. (18) and using the continuity equation to exclude the electron number density, after some straightforward manipulations we obtain

$$\chi\frac{Tl}{\mathbf{M}^2}\frac{dV_x}{dx} = (1 + A)^2 V_x^2 + \text{const}\, V_x + \frac{T}{\mathbf{M}^2}, \tag{21}$$

where $\chi \approx 0.74$. Now we prove that this equation have solutions resembling the shock waves.

If instead of the integration constant const we introduce two constants, V_a and V_b such that

$$V_a V_b = \frac{T}{(1 + A)^2 \mathbf{M}^2} \tag{22}$$

and

$$\text{const} = -(1 + A)^2 (V_a + V_b),$$

this equation takes the form

$$\chi\frac{Tl}{M^2}\frac{dV_x}{dx} = (1 + A)^2 (V_x - V_a)(V_x - V_b). \tag{23}$$

Without the loss of generality, we can assume that $V_a < V_b$. Taking into account that l is negative, we see that this equation has a monotone increasing solution approaching V_a and V_b as $x \to -\infty$ and $x \to +\infty$, respectively.

Although the general system of equations (7)–(12) appears to be more elegant when written in terms of electron velocities, in this region, where the ions are drastically accelerated, it is more convenient to rewrite Eq. (22) in terms of ion velocity and ion Mach number,

$$\Delta_v \frac{dV}{dx} = (V_b - V)(V - V_a), \tag{24}$$

where

$$\Delta_v = \chi\frac{I_e}{I_i}\frac{T}{\mathbf{M}^2} l_i, \qquad l_i = V_{i0}\tau,$$

V is the ion velocity; all variables are dimensionless except x and characteristic lengths l_i and Δ_v , which are measured in cm. The solution to Eq. (24) is

$$V = \frac{V_a \exp(-\tilde{x}) + V_b \exp(\tilde{x})}{2\cosh(\tilde{x})},$$

where $\tilde{x} = (V_b - V_a)(x - x_0) / (2\Delta_v)$.

Let us obtain the relationship between the velocities upstream and downstream of the discontinuity. For simplicity, chose the values at the point b for normalization. Then $V_b = 1$ for both ion and electron velocities, and T = 1. From Eq. (22) rewritten in terms of ion velocities, we obtain

$$V_a = \frac{1}{\mathbf{M}^2}.$$

This equation is an analogue of the Rankine-Hugoniot relations for shock waves. We observe that the velocities upstream and downstream are different if $\mathbf{M} > 1$, the difference vanishes as $\mathbf{M} \to 1$.

From Eq. (25) it follows that

$$\frac{\varepsilon_b}{\varepsilon_a} = M^4,$$

where $\varepsilon_{a,b}$ are ion kinetic energies upstream and downstream the discontinuity.

The thickness of the acceleration zone can be estimated as

$$\Delta_{\text{acc}} \approx (V_b - V_a)\left(\frac{dV}{dx}\right)^{-1} = \frac{3l}{\mathbf{M}^2 - 1},$$

where the derivative is calculated at the point of inflection on the ion velocity profile. The quantity Δ_{acc} can be considered as a rather rough estimate because for such small scales the hydrodynamic approach fails to be applicable. However, the results obtained from hydrodynamic equations in similar situations usually seem to be quite reasonable (see, e.g., [17]). In our case $\Delta_{\text{acc}} \approx 3.5$ mm in good agreement with experimental results shown in Fig. 1.

Finally, we discuss the procedure for obtaining a solution describing the plasma flow within the thruster channel from anode to exhaust.

The first approximation of the solution is obtained as follows (compare with the similar problem for shocks, see, e.g., [17]). Although for the low-power thruster considered, the acceleration zone approximately coincides with the exit plane, this is not the case for other SPT (see, e.g., [12]). Here we assume that in the near-cathode region the gradients of the plasma parameters are small enough for electron viscosity effects to be negligible. In this case the plasma flow both in the near-anode and near-cathode regions can be described by the same equations. Suppose that we know all parameters of plasma flow in the vicinity of the anode and near the exhaust. Then integrating Eqs. (8), (14) and (15) from anode to exhaust and vice versa, we obtain the fragments of the solution describing the large scale plasma flow. Next, we find a point within the channel, where the analogue (25) of the Rankine-Hugoniot relations is satisfied. It can be easily shown that this point exists and is unique. At this point the acceleration zone is located. At the first approximation, the thickness of the zone is neglected, i.e., we obtain the solution with a discontinuity. If required, the finite thickness of the zone can be taken into account in the second approximation [16, 17] and as a result a smooth solution can be obtained. Analytical methods for obtaining such solutions are outlined in [16]. However, since the corresponding formulas are cumbersome, usually the numerical methods are used rather than analytical.

5. Conclusions

We develop a new stationary hydrodynamic model for Hall thruster. Our model is more general than the models suggested previously.

The model equations are analyzed using the asymptotic method of multiple scales. The analysis shows that under the typical conditions for SPT operation a "discontinuity" can develop in the plasma flow within the SPT channel. Within such a "discontinuity" a considerable acceleration of ions occurs. The thickness of the "discontinuity" can be determined by the electron viscosity.

It is worth noting that in some previous papers (see, e.g., [6, 7]) the attempts were made to find a solution without a discontinuity. However, if such a solution does exist from the formal viewpoint, the question remains to be answered whether this solution is typical for real thrusters. To answer this question, additional experimental measurements of the plasma flow parameters within the SPT channel are required. The results of the measurements combined with numerical calculations can shed some light on the role and nature of different dissipative mechanisms (for example, the role of anomalous thermoconductivity, dissipation due to collisions of the thermal electrons with the walls of the channel, etc.). Another question, which is closely related to the previous one, is concerned with the stability of the solutions with the discontinuity and without it. This question will be addressed in the next paper.

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