

Numerical simulation of two-dimensional, non-steady, ideal magnetogasdynamics equations

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ABSTRACT

The methodology to numerically solve two-dimensional, non-steady, ideal magnetogasdynamics equations (MGD), together with the results obtained with a computational code developed by the authors, are presented in this paper. This research is an intermediate stage towards achieving a complete description of the inner plasma flow of an ablative pulsed plasma thruster (APPT). The magnetogasdynamics system is described by means of continuity, momentum, energy, and state equations. This partial differential equations system can be expressed as an hyperbolic system using the eight waves technique introduced by Powell, while the eigenvectors are normalized to avoid numerical troubles. The numerical approach is based on using an approximate Riemann solver as a high resolution technique, together with a TVD scheme. To improve the numerical results a modification of the original TVD scheme is introduced. Finally, the computational code is validated by comparing the numerical results with published data from other authors in the scientific bibliography.

INTRODUCTION

The main objective of this work is to present the results obtained by means of a software developed to solve the non-steady, two-dimensional equations of the ideal magnetogasdynamics (MGD). Throughout the literature, papers can be found which use or describe numerical codes to simulate the flow inside of plasma thrusters. For example, two-dimensional codes have already been developed by Toki *et al.*,¹ and Ao and Fujiwara.² The effects of the geometry in the performances have been studied by LaPointe³, and Mikellides and Turchi⁴ using the codes MACH2 and MACH3. However, none of this developments implement high-resolution techniques having been used with success in gasdynamics for capturing discontinuities. The present investigation explores the ability of these techniques to simulate MGD flows. It is important to highlight that Sankaran and Choueiri^{5,6} use a method similar to what is implemented in the present research, although the numerical techniques show some differences. We utilize a TVD technique instead of their LED scheme as developed by Jameson.⁷

Electric propulsion is presently being used to propel satellites, however there are developments aiming to send electrically propelled ships to the Moon and Mars.⁸ The main advantage of electric propulsion is that it reduces the launching weight considerably and accordingly its costs. Considering the underlying physics of the thrust producing phenomenon, electric thrusters can be classified in three categories: electrothermal, electrostatic and electromagnetic. Pulsed plasma thrusters, like the one being presently developed in Argentina,⁹ are included in the third category.

The equations that governs the magnetogasdynamics flow are continuity, balance of momentum, balance of energy and magnetic induction together with a state equation. These equations when written in conservative form define an hyperbolic-parabolic system of equations.⁵ The hyperbolic terms represent the convective effects and the parabolic ones the diffusive effects. If the parabolic terms are neglected the resulting system corresponds to the ideal magnetogasdynamics. The implemented numerical technique consists of an approximate Riemann solver that allows to evaluate the variables inside each cell by means of the variation of the flows through the contour of this cell. Due to the accurate performances shown by the TVD technique proposed by Yee *et al.*¹⁰ in the simulation of the supersonic flow of gases,¹¹ this scheme is

chosen to numerically evaluate magnetogasdynamics flows. To obtain eight real eigenvalues, the eight-wave technique introduced by Powell is used.¹² The eigenvectors are normalized according to the works of Zarachay *et al.*,¹³ Roe and Balsara,¹⁴ and Bodgan.¹⁵ This methodology has already been successfully proven to solve the unsteady, one dimensional ideal and real magnetogasdynamics equations.¹⁶

Finally, with the objective of verifying the correct behavior of the software, the Riemann problem introduced by Brio and Wu¹⁷ is considered. as a “benchmark”. To represent a two-dimensional flow the mesh is rotated by a 18.4° angle with respect to the main flow direction.

EQUATIONS OF THE IDEAL MAGNETOGASDYNAMICS

The non-dimensional equations of the ideal magnetogasdynamics considering two-dimensional and unsteady flows, written in conservative form are expressed as:

$$\frac{\partial \underline{U}}{\partial t} + \frac{\partial \underline{F}}{\partial x} + \frac{\partial \underline{G}}{\partial y} = \underline{0} \quad (1)$$

where \underline{U} is the vector of state variables (conservative variables)

$$\underline{U} = \left(\rho, \rho u_x, \rho u_y, \rho u_z, B_x, B_y, B_z, e \right)^T \quad (2)$$

and the hyperbolic fluxes are:

$$\underline{F} = \begin{pmatrix} \rho u_x \\ \rho u_x^2 - B_x^2 + p + \frac{1}{2} B^2 \\ \rho u_x u_y - B_x B_y \\ \rho u_x u_z - B_x B_z \\ 0 \\ u_x B_y - u_y B_x \\ u_x B_z - u_z B_x \\ \left(e + p + \frac{1}{2} B^2 \right) u_x - (\underline{B} \cdot \underline{u}) B_x \end{pmatrix} \quad \underline{G} = \begin{pmatrix} \rho u_y \\ \rho u_x u_y - B_x B_y \\ \rho u_y^2 - B_y^2 + p + \frac{1}{2} B^2 \\ \rho u_z u_y - B_z B_y \\ u_y B_x - B_y u_x \\ 0 \\ u_y B_z - B_y u_z \\ \left(e + p + \frac{1}{2} B^2 \right) u_y - (\underline{B} \cdot \underline{u}) B_y \end{pmatrix} \quad (3)$$

ρ indicates the density; u_x u_y u_z are the velocity components; B_x B_y B_z represent the components of the magnetic field vector; p is the pressure; and $B^2 = B_x^2 + B_y^2 + B_z^2$. The total energy in non-dimensional form is expressed as:

$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \left(u_x^2 + u_y^2 + u_z^2 \right) + \frac{1}{2} \left(B_x^2 + B_y^2 + B_z^2 \right) \quad (4)$$

where γ is the specific heats ratio.

The equations system (1) can be written in quasi-linear form:

$$\frac{\partial \underline{U}}{\partial t} + [A_c] \frac{\partial \underline{U}}{\partial x} + [B_c] \frac{\partial \underline{U}}{\partial y} = \underline{0} \quad (5)$$

$$[A_c] = \frac{\partial \underline{F}}{\partial \underline{U}} \quad [B_c] = \frac{\partial \underline{G}}{\partial \underline{U}} \quad (6)$$

where $[A_c]$, $[B_c]$ are the Jacobian fluxes; the letter “c” indicates that the derivation has been carried out with respect to the conservative state variables. However, the Jacobian fluxes have a simpler form when expressed as a function of the primitive variables (W):

$$\underline{W} = (\rho, u_x, u_y, u_z, B_x, B_y, B_z, p)^T \quad (7)$$

Consequently, Eq. (5) can be expressed in terms of the primitive variables:

$$\frac{\partial \underline{W}}{\partial t} + [A_p] \frac{\partial \underline{W}}{\partial x} + [B_p] \frac{\partial \underline{W}}{\partial y} = \underline{0} \quad (8)$$

The transformation rule is:

$$[A_p] = \left(\frac{\partial \underline{W}}{\partial \underline{U}} \right) [A_c] \left(\frac{\partial \underline{U}}{\partial \underline{W}} \right) \quad [B_p] = \left(\frac{\partial \underline{W}}{\partial \underline{U}} \right) [B_c] \left(\frac{\partial \underline{U}}{\partial \underline{W}} \right) \quad (9)$$

Because the Jacobian flux possesses a null eigenvalue, it is not possible to use a Riemann solver based on an eight waves system. It is important to highlight that both formulations, in primitive variables as well as in conservative variables, are equivalent, consequently the null eigenvalue appears in both formulations. To solve this inconvenient, alternative Jacobian fluxes were implemented (A'_p, B'_p), as presented by Powell.¹²

The eigenvalues of the matrix A'_p are:

$$\lambda_e = u_x; \quad \lambda_a = u_x \pm c_a; \quad \lambda_f = u_x \pm c_f; \quad \lambda_s = u_x \pm c_s; \quad \lambda_d = u_x \quad (10)$$

The Alfvén velocity and the slow and fast magnetosonic velocities are expressed respectively as:

$$c_a = \frac{|B_n|}{\sqrt{\rho}} \quad c_{f,s}^2 = \frac{1}{2} \left[\frac{\gamma p + B^2}{\rho} \pm \sqrt{\left(\frac{\gamma p + B^2}{\rho} \right)^2 - 4 \frac{\gamma p B_n^2}{\rho^2}} \right] \quad (11)$$

where the positive sign corresponds to the fast magnetosonic wave. Finally, following the proposal of Zachary,¹³ the eigenvectors have been normalized to avoid degeneration of the system.

NUMERICAL METHOD

The system of equations (1) is solved using an approximate Riemann solver together with an explicit scheme. The numerical flows are evaluated by means of the Harten-Yee TVD technique.¹⁰ This technique allows to correctly capture discontinuities and to achieve a second order approach where the solution is a smooth one. This method is used to evaluate the numerical flows in all cells. A concept outline is shown in Figure 1.

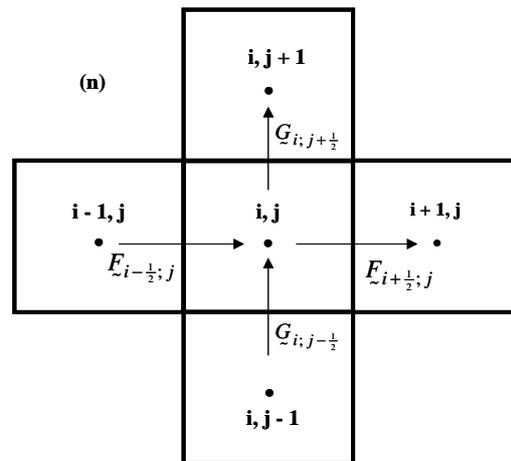


Figure 1. Two-dimensional numerical method.

The explicit TVD scheme can be expressed as:

$$\underline{U}_{ij}^{n+1} = \underline{U}_{ij}^n - \frac{\Delta t}{\Delta x} \left(\overline{F_{i+\frac{1}{2};j}^n} - \overline{F_{i-\frac{1}{2};j}^n} \right) - \frac{\Delta t}{\Delta y} \left(\overline{G_{i;j+\frac{1}{2}}^n} - \overline{G_{i;j-\frac{1}{2}}^n} \right) \quad (12)$$

The functions that determines the numerical fluxes are defined as:

$$\overline{F_{i+\frac{1}{2};j}^n} = \frac{1}{2} \left(F_{i+1}^n + F_i^n + \left(\sum_m R_{i+\frac{1}{2}}^m \Phi_{i+\frac{1}{2}}^m \right)^{(n)} \right) \quad (13)$$

where $[R]$ is the matrix formed by the right eigenvectors of $[A_c]$. The dissipation function is expressed as:

$$\Phi_{i+\frac{1}{2}}^m = (g_{i+1}^m + g_i^m) - \sigma (\lambda_{i+\frac{1}{2}}^m + \gamma_{i+\frac{1}{2}}^m) \alpha_{i+\frac{1}{2}}^m \quad (14)$$

Due to the accurate results found for the one-dimensional flows,¹⁶ the following ‘‘limiter’’ function is used:

$$g_i^m = \text{sgn}(\lambda_{i+\frac{1}{2}}^m) \max \left\{ 0, \min \left[\sigma_{i+\frac{1}{2}}^m |\alpha_{i-\frac{1}{2}}^m|, \sigma_{i-\frac{1}{2}}^m \frac{\text{sgn}(\lambda_{i+\frac{1}{2}}^m)}{2} \alpha_{i-\frac{1}{2}}^m \right] \right\} \quad (15)$$

with

$$\sigma(z) = \begin{cases} |z| & |z| \geq \varepsilon \\ \frac{1}{2\varepsilon} (z^2 + \varepsilon^2) & |z| < \varepsilon \end{cases} \quad (16)$$

and

$$\gamma_{i+\frac{1}{2}}^m = \begin{cases} \frac{1}{\alpha_{i+\frac{1}{2}}^m} (g_{i+1}^m - g_i^m) & \alpha_{i+\frac{1}{2}}^m \neq 0 \\ 0 & \alpha_{i+\frac{1}{2}}^m = 0 \end{cases} \quad (17)$$

In order to simplify the calculation process, the primitive variables are used to obtain:

$$\alpha^m = \underline{L}_p^m \cdot (W_{i+1} - W_i), \quad (18)$$

\underline{L}_p^m being the left eigenvector of the matrix $[A_p]$ associated to the m -th wave.

RESULTS

In this section the numerical results obtained considering the Riemann problem¹⁷ are presented. Initially, there is a discontinuity that separates two constant states, one left and another right. These states are determined by their respective initial conditions. With the objective of verifying the correct operation of the developed software, the mesh is rotated by a 18.4° angle with respect to the longitudinal axis of the flow. Then, the software is forced to simulate a two-dimensional flow. The problem geometry is shown in Figure 2. The mesh have 1000 nodes in the horizontal direction and 350 nodes in the vertical direction with

$$\Delta x = \Delta y = 0.001 \quad (19)$$

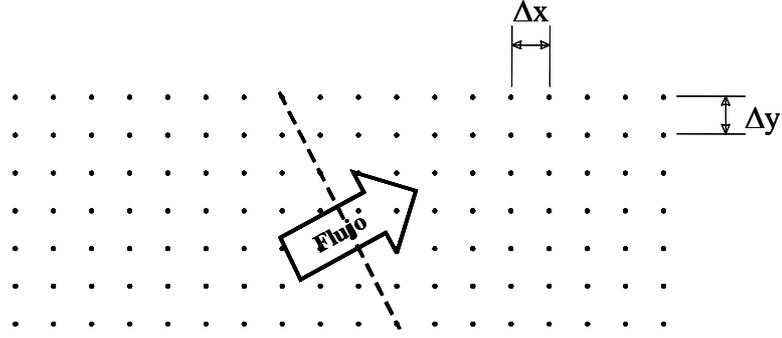


Figure 2. Two-dimensional Riemann problem.

Gasdynamics benchmark

In this benchmark, the software ability to solve problems with zero magnetic field is evaluated. The initial conditions are the following:

$$\begin{aligned} W_l &= (1.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1.0)^T \\ W_r &= (0.125, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.1)^T \end{aligned} \quad (20)$$

In Figures 3, 4, 5 and 6, the new results are compared with those obtained for one-dimensional flows.

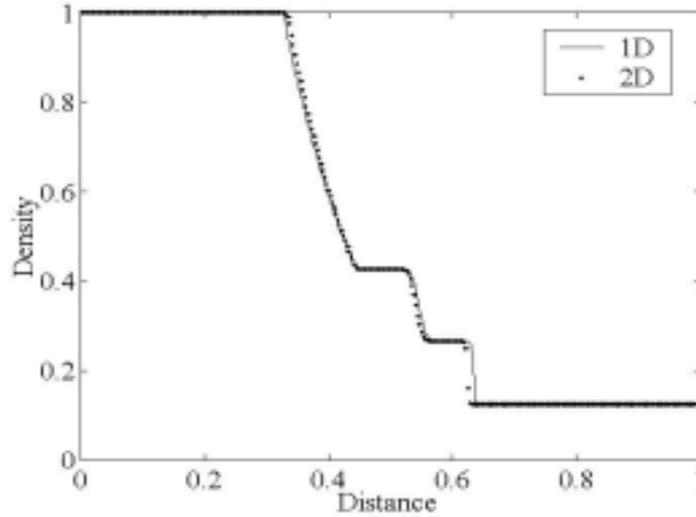


Figure 3: density as a function of distance along the flow direction.

To obtain the solution for the gasdynamics equations the software uses Eq. (12) without considering the numerical fluxes corresponding to the magnetic field, because these fluxes introduce numerical errors.

Sensitivity Analysis

In order to minimize the interferences introduced by the artificial damping as represented by the dissipation function, we have carried out simulations with different values of ε in Eq. (16). Some of the obtained results are shown in Figure 10. All results were produced with a CFL < 0.5. It is possible to see that for $\varepsilon \leq 0.01$ the numerical results present oscillations and the solution moves away from the “real solution” as the time evolves. When $\varepsilon = 1$ the results are satisfactory, the two-dimensional solution approaches the one-dimensional solution and the interferences are reduced and located in a small interval. In order to improve the technique, ε is considered as a variable, being evaluated as an average of the eigenvalues for each location and time. The best solution is found to be reached for variable ε , as shown in Figure 10.

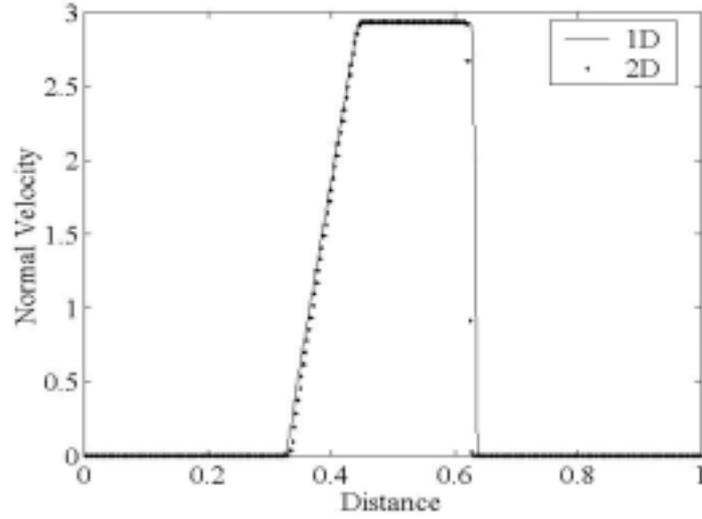


Figure 4: Normal velocity as a function of the longitudinal distance.

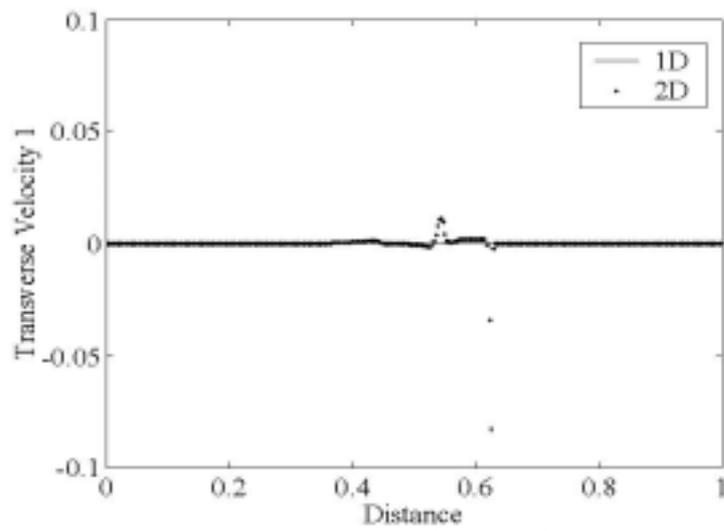


Figure 5: Transverse velocity 1 as a function of the longitudinal distance.

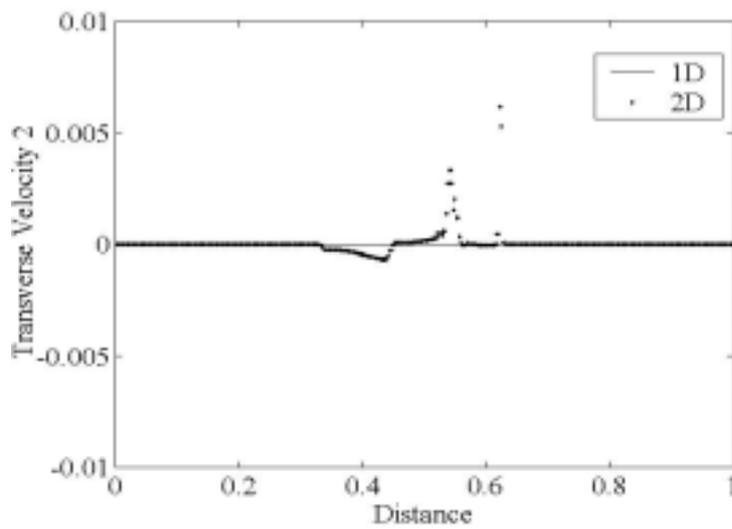


Figure 6: Transverse velocity 2 as a function of the longitudinal distance.

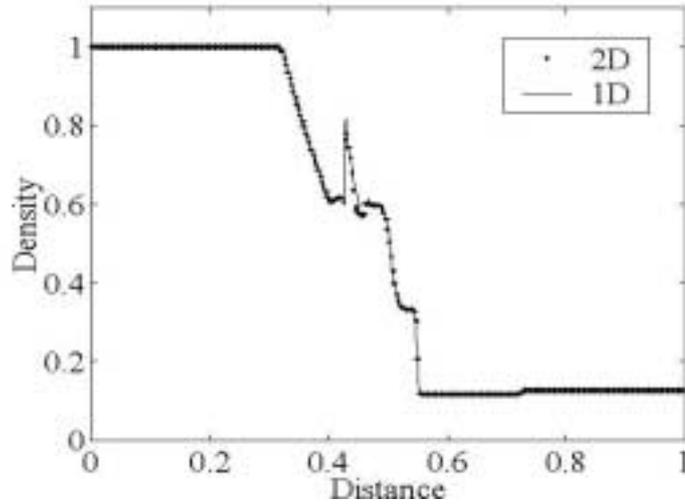
Magnetogasdynamics benchmark

The results presented in this section are obtained for the Riemann problem as proposed by Brio and Wu.¹⁷ This is a benchmark used by the scientific community to evaluate the behavior of numerical techniques and the related software. For magnetogasdynamics flows, this benchmark is named coplanar Riemann problem.

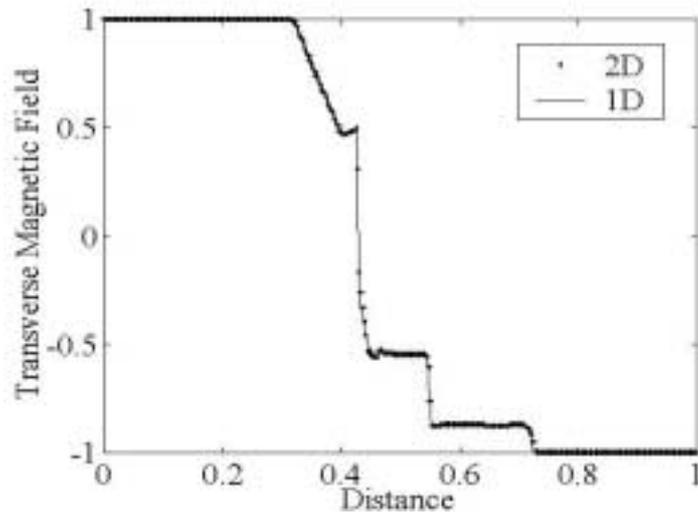
The initial conditions used in the simulation are:

$$\begin{aligned} W_l &= (1.0, 0.0, 0.0, 0.0, 0.0, 0.75, 1.0, 0.0, 1.0)^T \\ W_r &= (0.125, 0.0, 0.0, 0.0, 0.0, 0.75, -1.0, 0.0, 0.1)^T \end{aligned} \quad (21)$$

Figures 7, 8 and 9 present the solution for one-dimensional and two-dimensional flows.



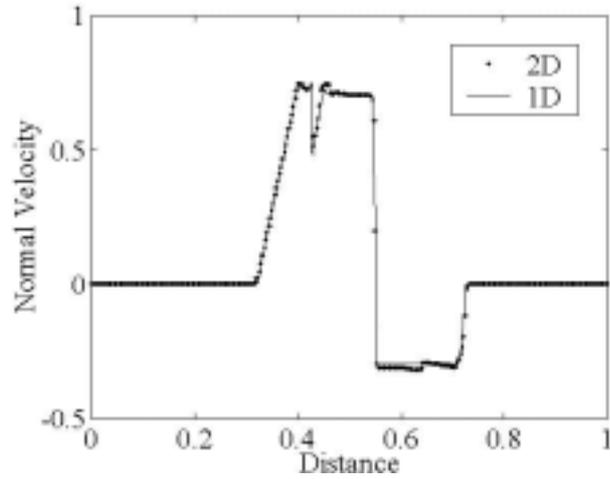
Figures 7: Density as a function of the distance along the flow direction.



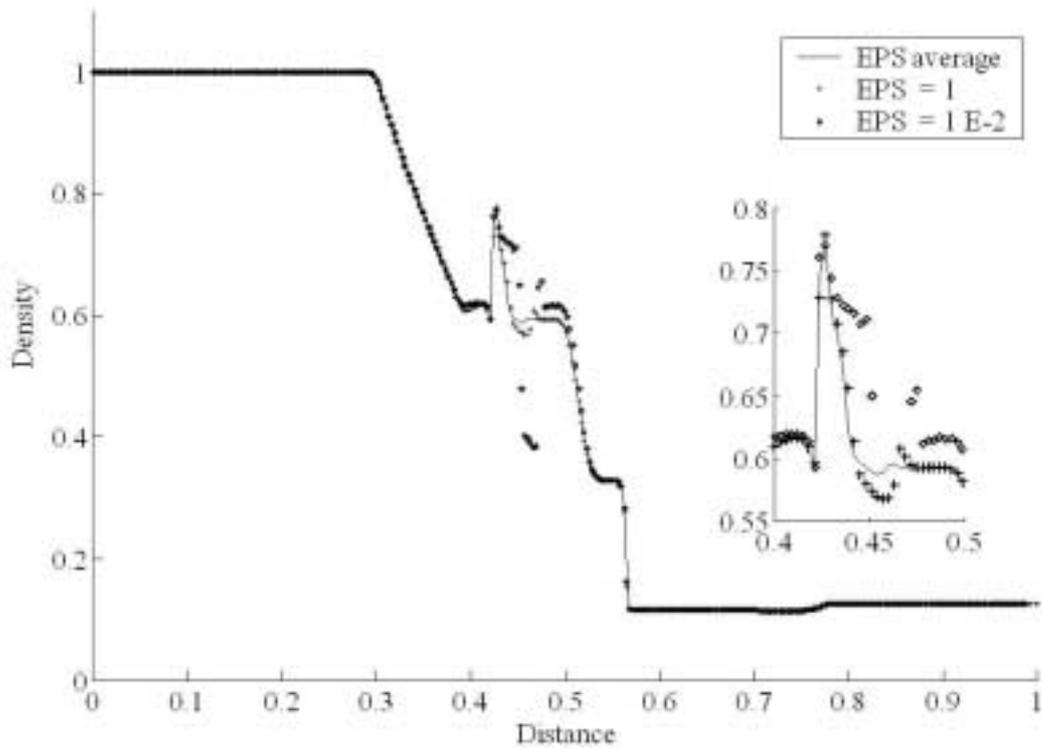
Figures 8: Transverse magnetic field as a function of the distance along the flow direction.

In Figure 7 it is possible to distinguish, from left to right, firstly the expansion wave that travels toward the left, then a compound wave, a contact discontinuity, a slow shock wave and the expansion wave that travels toward the right. To obtain these results it was necessary to use 1000 steps of time with a CFL number equal to 0.4. It is important to note that this simulation took around 96 hours computation time in a PC-Pentium IV 1.5 GHz, with 256 Mb CPU RAM.

From the figures, one can observe that the numerical methodology is able to satisfactorily capture all the discontinuities, and the numerical values obtained in this research closely agree with the results published by Brio and Wu¹⁷ and Bodgan.¹⁵



Figures 9: Normal velocity as a function of the distance along the flow direction.



Figures 10: Influence of the parameter ε .

Analysis of the dissipation functions

The main objective of this section is to study the influence of the numerical dissipation functions. It can be seen from Eq. (13) that the fluxes associated to each conservative variable split in two parts, the first one corresponds to real or “physical” values while the second one introduces the numerical dissipation. The influence of the dissipation on density and magnetic field for the coplanar Riemann problem is shown in Figures 11 and 12.

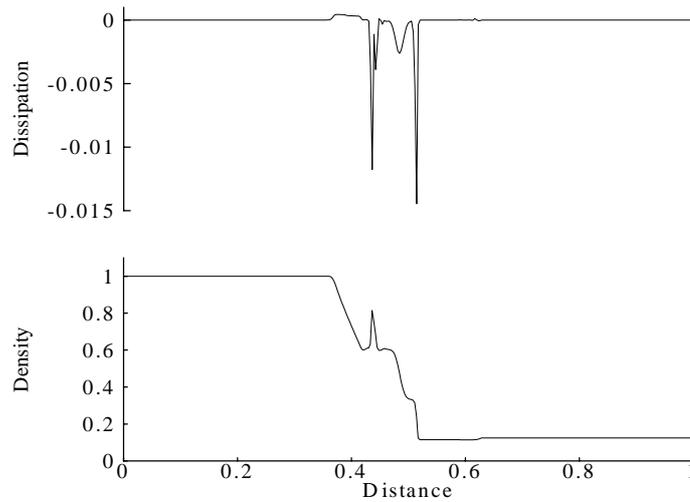


Figure 11: Influence of the parameter dissipation functions by the density.

From the above pictures it is possible to conclude that the Harten-Yee TVD technique works correctly, since where the solution is smooth the dissipation function is null. On the other hand, in locations where a wave evolves the dissipation is readily introduced, thus avoiding solution oscillations while appropriately reproducing the physical phenomenon.

CONCLUSIONS

The obtained numerical results for the gasdynamics as well as for the magnetogasdynamics equations, compare satisfactorily with the solutions published by other authors. The implementation of an averaged value for ε considerably improves the solution, specially in regions where very small interferences take place due to the numerical damping introduced through the dissipation functions. The employed Harten-Yee TVD technique has shown to be able to simulate gas and MGD flows. However, the calculation is highly time consuming, so further improvement remain to be done in this regard. In spite of this, the new software has been shown to be a robust one through validation with a very demanding benchmark.

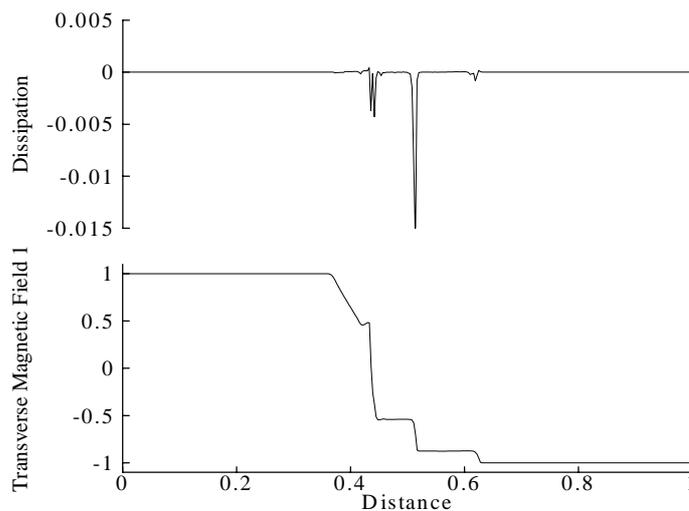


Figure 12: Influence of the parameter dissipation functions by the transverse magnetic field.

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