# SOLUTIONS WITH SHOCKS FOR FLUID MODEL OF STATIONARY PLASMA THRUSTER

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**Abstract**

In this papper we study the weak solutions to the simplified system of equations resulting from the fluid model of a "stationary plsama truster". Because of the nonlinearity the classical solutions exist in general only for a limited period of time, whose length depends on the initial data. The notion of a weak solution and in particular a solution with shocks is introduced here in a different manner from what is known from the theory of nonlinear hyperbolic systems of conservation laws. The difference comes from the fact that the ion fluid in the plasma thruster is practically collisionless which is in contrast to ordinary gas-dynamics, where the mean free path is assumed to be much smaller then the characteristic macroscopic dimension. Therefore we allow for ion velocity to become a multivalued function, whereas the ion density can be not only discontinuous but it can even contain the Dirac delta measures.We demonstrate also some examples of numerical solutions with "shocks".

## 1 Introduction

This paper is devoted to the question of mathematical modeling of the SPT and, in particular, to examine the possibility of introducing the notion of physically meaningfull weak solutions. Such a definition seems to be necessary for the construction of numerical schemas allowing for the assessment of global in time solutions when starting from arbitrary initial data. We confine our attention to the so called fluid models of the plasma discharge [1,3,4]. Other models as, for instance kinetic or hybrid models will not be considered here. The system of equations describing such a model is highly nonlinear and one can not expect that the classical, smooth solutions exist for period of time long enough.Typically the gradient catastrophe appears - the gradient becomes infinite and the classical solution ceases to exist. This fact was already noticed in the case of equations of an ideal gas by G.B.Riemann in sixties of the nineteen century. In order to cope with such a pathology he proposed to introduce some more general solutions with possible discontinuities - "shock waves". In reality this discontinuities have some non vanishing thickness - of the order of a few mean path of the particles which constitutes the considered fluid. The most common approach to introduce such generalized idealized, discontinuous (called also weak) solutions, starts with some "physically relevant" system of conservation laws, i.e. the system whose left hand side is a divergence of some set of vector fields depending on the unknown functions. The other, seemingly redundant conservation laws (less physically relevant), as for example conservation of entropy in gas-dynamics, must be converted to inequalities only, which are used then to distinguish between the physical and unphysical shocks. Having this divergent form of the equations it is easy then to treat all the derivatives in the distributional sense. It appears that in the theory developed in this paper and in a more detailed way in [15] the classical notion of a weak solution is not very usefull and the main reason for it follows from the fact that the mean free path for the ion - ion collision is many times larger then the size of the thruster. In such a case we must admit rather multivalued functions as possible ion velocity distributions even if the initial velocity was assumed to be single valued. It appears also, that the crucial role in the paper is played by the "regularization procedure" which is based on the Debye length. Applying this regularization we can transform the considered system of equations into an ordinary differential equation in some Banach space. In order to do this we

use the Lagrange coordinates. However, as the inverse transformation is not always well defined, we have to define the solution in Eulerian coordinates in an appropriate way.This will lead us to the notion of the "weak solution". It may be happen after some time we can have ions with different velocities, which initialy were in different places. This would not be true, obviously in the case of a dense fluid. Therefore in order to find the density of ions at some point in the Eulerian coordinates we have to take into account all coexisting there ion populations.

In the next section we give the formulation of our transient 1-D fluid model. Then we discuss possible simplifications. First of all we discuss briefly the model based on the plasma neutrality assumption leading to the nonlinear hyperbolic system. Then we discuss further simplifications; after neglecting the electron pressure one arrives at so called hypersonic approximation which appears to be mathematically incorrect. To regularize this model one has to go beyond the neutrality assumption. The regularized system is mathematically correct but has only local in time solutions. The regularized hypersonic approximation exhibits the role which is played by Eq. (2.8) for the electric field, i.e. how it influences the differentiability of various terms in the system. Because of nonlinearity all systems considered up till now have, in general, only local in time solutions. The numerical simulations suggest that in those cases when the evolution of plasma is expected to be violent as it should be in the initial stage after switching on the thruster, the regular solution ceases to exists and one should search for a weak solution.

## 2 Governing equations

At first like in most fluid and hybrid models, the electric neutrality will be assumed in the whole discharge channel. This is generally reasonable with the exception perhaps of the region close to the channel walls, where the Debye sheath exists. Later on, we will however take into account the Poisson equation, determining the electric field from the difference of the charge density. This will be done in an approximate way through some singular perturbation at the limit of vanishing electron mass. Ions are assumed to be insensitive to the magnetic field, since their Larmor radius exceeds the dimensions of the device. The motion of ions is thus considered as purely axial. The ion velocity dispersion is neglected (cold fluid approximation) which is justified by the fact that most of ions have been created and have low velocities before they enter the acceleration zone. Usually, the neutrals are assumed to have constant axial velocity. This can be justified if they perform only specular elastic reflection on the walls and if the ionization is not present. Otherwise one can demonstrate that the average velocity should be growing with the distance from the anode. For this reason, in this short presentation the equation for neutrals will not be discussed and in the following we will assume that the density of neutrals is a given function of time and x. However, in this section for the sake of completeness we will mention this simplest and not physically correct version with a given and constant flow velocity.

Under above assumptions, the plasma is described by the following equations:

- neutral continuity equation:

$$\frac{\partial N_a}{\partial t} + V_a \frac{\partial N_a}{\partial x} = -\beta\, N_a n_e, \tag{2.1}$$

where $N_a$ is the neutral density, $V_a$ the neutral velocity, $n_e$ the electron density, $n_i$ the ion density and $\beta = \beta(T_e, E_{ke})$ the ionization rate which is a function of the electron temperature $T_e$ and of the drift kinetic energy of electrons $E_{ke} = \frac{1}{2} m_e (V_{ex}^2 + V_{e\theta}^2)$.

- ion continuity and momentum equation:

$$\frac{\partial n_i}{\partial t} + \frac{\partial\,(n_i V_i)}{\partial x} = \beta\, N_a n_e, \tag{2.2}$$

$$\frac{\partial\,(n_i V_i)}{\partial t} + \frac{\partial\,\left(n_i V_i^2\right)}{\partial x} = \frac{n_i e\, E}{m_i} + \beta\, n_a n_e V_a, \tag{2.3}$$

where $V_i$ is the ion velocity, $m_i$ the ion mass and $E$ the axial electric field.

- electron continuity, momentum and energy equations

$$\frac{\partial n_e}{\partial t} + \frac{\partial\,(n_e V_e)}{\partial x} = \beta\, N_a n_e, \tag{2.4}$$

$$\frac{m_e}{m_i}\left(\frac{\partial}{\partial t}(n_e V_{ex}) + \frac{\partial}{\partial x}(n_e V_{ex}^2)\right) + \frac{\partial}{\partial x}\left(\frac{kT_e}{m_i}\, n_e\right) = -\frac{n_e e\, E}{m_i} + n_e\,(\tilde{\omega}_B V_{e\theta} - \tilde{\nu}_m V_{ex}), \tag{2.5}$$

$$\frac{m_e}{m_i}\left(\frac{\partial}{\partial t}(n_e V_{e\theta}) + \frac{\partial (n_e V_{ex} V_{e\theta})}{\partial x}\right) = -n_e \tilde{\omega}_B V_{ex} + n_e \tilde{\nu}_m V_{e\theta} \tag{2.6}$$

$$\frac{\partial}{\partial t}\left(n_e E_k + \frac{3}{2} n_e k T_e\right) + \frac{\partial \left[n_e V_{ex} E_{ke} + n_e V_{ex} \frac{5}{2} k T_e\right]}{\partial x} = Q_{\text{Joules}} - Q_{\text{ioniz}} - Q_{\text{wall}} \tag{2.7}$$

with $Q_{\text{Joules}} = -V_{ex} n_e e\, E$, $Q_{\text{ioniz}} = \gamma_{ion} e U_{ion} \beta\, N_a n_e$ and $Q_{\text{wall}} = \nu_{ew} n_e [E_{ke} + 2kT_e + (1-\overline{\tau}) e\phi_w$, where $V_{e\theta}$ and $V_{ex}$ are the electron velocity components in the azimuthal and axial direction. We have denoted here

$$\tilde{\nu} = \frac{m_e}{m_i}\nu_m, \quad \tilde{\omega}_B = \frac{m_e}{m_i}\omega_B,$$

where $\nu_{ew}$ is the inelastic electron wall collision frequency, $\nu_m$ is the total momentum transfer collision frequency for electrons (which accounts for electron-atoms and electron-wall collisions), $\omega_B$ is the Larmor frequency, $U_{ion}$ is the ionization potential of xenon, $\gamma_{ion}$ is the factor of effective ionization cost, $\overline{\tau}$ is the mean electron secondary emission (which is a function of $T_e$) and $\phi_w$ is the potential drop across the electric sheath at the ceramic walls.

- the "Poisson" equation

$$\varepsilon_0 E_{,x} = e(n_i - n_e) \tag{2.8}$$

determining the electric field from the charge densities.

The Poisson equation creates often the computational problems, mainly in those cases when $|n_i - n_e| \ll n_i$. This in fact is the case of our plasma. The ratio $(n_i - n_e)/n_i$ is of the order of $10^{-5}$. In such cases very small error in determination of negative and positive charge densities highly influences the computed electric field which in turn influences the plasma dynamics and hence again increases the indeterminacy of $n_i$ and $n_e$. To avoid such problems one usually assumes plasma quasi neutrality, $n_i = n_e$, in all other equations except of Eq. (2.8) which must be rejected. Then, after passing to the limit $m_e/m_i = 0$, the electric field follows from Eq. (2.5) as the "reaction of the system" to the constraint $n_i = n_e$.

## 2.1 Simplifications

Applying the electric neutrality assumption, $n_i = n_e$ and charge conservation we can eliminate $V_{ex}$ by noticing that $I = (n_i V_i - n_e V_{ex})e$ where $I$ is the total current density. $I(t)$ satisfies in this approximation $\frac{\partial I}{\partial x} = 0$. Thus $I = I(t)$ can depend only on time.

The electric field can also be eliminated using equation (2.5) and (2.6) where we neglect electron inertia. In this way we obtain $\omega_B V_{ex} + \nu_m V_{e\theta} \approx 0$ which allows for elimination of $V_{e\theta}$ and finally we arrive at the generalized Ohm law

$$\frac{e}{m_i} E = -\nu_{\text{eff}} V_{ex} - \frac{1}{n_e}\frac{\partial}{\partial x}\left(\frac{k T_e}{m_i} n_e\right), \tag{2.9}$$

where $\nu_{\text{eff}} = \frac{m_e}{m_i}\left(\nu_m + \frac{\omega_B^2}{\nu_m}\right)$. Further approximation is made by assuming that the RHS of equations (2.6) and (2.7) is represented by the difference of two large terms ([gain] and [loss]), which implies that the LHS of this equation can be neglected, thus giving

$$Q_{\text{Joules}} - Q_{\text{ioniz}} - Q_{\text{wall}}, \quad -(\beta N_a - \sigma_i)\left(\frac{5}{2} k T_e - E_{ke}\right) \approx 0. \tag{2.10}$$

This equation can serve for determination of the electron temperature as a function of other variables.

## 2.2 Hyperbolic approximation

After above simplification the final system of equations for the set of variables $N_a$, $n_i$ and $V_i$ is then given by

$$\frac{\partial N_a}{\partial t} + V_a \frac{\partial N_a}{\partial x} = -\beta\, N_a n_i, \tag{2.11}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i V_i)}{\partial x} = \beta\, N_a n_i, \tag{2.12}$$

$$\frac{\partial\left(n_i V_i\right)}{\partial t}+\frac{\partial\left(n_i V_i^2+n_i\frac{kT_e}{m_i}\right)}{\partial x}=\nu_{\text{eff}}\left(I-n_i V_i\right)+\beta\, N_a n_i V_a, \tag{2.13}$$

The electron temperature $T_e$ can be determined as the solution of the transcendental equation (2.10). The last equation contains the quantity $I(t)$ representing the density of the total current (i.e. electron + ion). The charge conservation implies that $I$ is not dependent on $z$ and may depend on $t$ only, so $I = I(t)$. In view of this remark, at every moment the discharge current is determined from the requirement of constant discharge voltage value $U_0$, i.e.

$$U_0=\int_0^L E\,dx, \tag{2.14}$$

where $E$ is defined by (2.9). Since it is assumed that $I$ depends only on $t$ then $I(t)$ can be expressed in terms of integrals from other functions.

It has to be pointed out that for the system constituted by Eqs (2.12) and (2.13), it is sufficient to pose one boundary condition at $x = 0$ since only one characteristic is entering the domain. At the exit plane, no boundary condition is needed since the ion flow is supersonic and all characteristics leave the domain (no information propagates backward). Note, however, that if the full energy equation was kept, one boundary condition on the electron temperature would be required in the exit plane.

For a given temperature $T(t, x)$ the system (2.11–2.13) is hyperbolic. Therefore the existence and uniqueness of local in time solutions relatively easily follows from the general theory [8]. An additional difficulty is caused here by the fact that $I(t)$ is computed as an integral involving the unknown functions. The problem becomes more delicate, however, when $T_e$ is computed from the implicit relation (2.10). It seems that depending on "material coefficients" in (2.10) there are various possibilities and it may happen that in some regions the system is not hyperbolic. Moreover, the system (2.12, 2.13) for given $N_a$, may become sometimes elliptic! Thus, from the mathematical point of view, it is probably easier to consider the system containing the full energy equation (2.7). It appears, however, that also in this case one encounters some difficulties – the system becomes degenerate on a certain hypersurface in the space of dependent variables.

Therefore **in the following**, to simplify our considerations, **we assume the electron temperature to be constant**.

# 3 Hypersonic approximation

The simplest system of equations is obtained, when in addition to the "electric neutrality assumption", also the pressure term is neglected. Under typical conditions the average electric field is at least one order bigger than the term involving pressure gradient [1,2]. One obtains in this case the following system of equations

$$\begin{aligned}
\frac{\partial N_a}{\partial t}+V_a\frac{\partial N_a}{\partial x}&=-\beta N_a n_i,\\
\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}(V_i n_i)&=\beta N_a n_i,\\
\frac{\partial V_i}{\partial t}+V_i\frac{\partial V_i}{\partial x}&=\frac{e}{m_i}E-\beta N_a\left(V_i-V_a\right),
\end{aligned} \tag{3.1}$$

where $eE = \nu_{\text{eff}} V_e$. The electron axial velocity $V_e$ can be computed from the relation for the total electric current

$$\frac{1}{e}I(t)=n_i V_i-n_e V_e.$$

Thus, by the neutrality assumption $n_e = n_i$ we have

$$eE=\nu_{\text{eff}}\left(\frac{I}{en_i}-V_i\right). \tag{3.2}$$

Here, the electric current is a function of $t$ only and it can be determined from the boundary condition for the potential

$$I(t)=\left(\frac{1}{e^2}\int_0^L\frac{\nu_{\text{eff}}}{n_i}\,dx\right)^{-1}\left\{U_0+\frac{1}{e}\int_0^L\nu_{\text{eff}}V_i dx\right\}. \tag{3.3}$$

Unfortunately, mathematically (3.1, 3.2) defines a weakly hyperbolic system, whose matrix is not diagonalizable and as it can be demonstrated [5], the Cauchy and, similarly, the initial boundary value problem is incorrectly posed.

On the other hand the numerical solutions of the system (3.1 – 3.3) seem to reflect quite well the experimental measurements [1]. Obviously, numerical solution represents always somewhat regularized problem because the finite grid is used. The question therefore arises whether it is possible to find a better, more physical, way of regularization of our system and explain why for a coarse grid the numerical solution is physically correct. It is well known that in the case of first order weakly hyperbolic systems (i.e. when the matrix $A$ of the system $U_t + AU_x = f(U,x,t)$ has real eigenvalues but it is not diagonalizable) – the solvability depends strongly on the right hand side, i.e. on $f(U,x,t)$. Comparing the formula (3.3) with a more fundamental Poisson equation $E_{,x} = \frac{e}{\varepsilon_0}(n_i - n_e)$ one notices that the reason for Hadamard instability follows from applying too bad approximation (3.3) for the electric field. The Poisson equation gives better differentiability of $E$ with respect to $x$ which is just needed to regularize system (3.1 – 3.3).

## 3.1 Regularization

Regularization is based on the fact that, due to the large mass difference between electrons and ions (in our case $\frac{m_e}{m_i} \approx 10^{-5}$ and therefore we assume $\frac{m_e}{m_i} = 0$ in our considerations), the electron density distribution $n_e(t,x)$ is much more smooth then the ion density $n_i(t,x)$. Similarly, Poisson equation leads to the electric field which is smoother than that given by (3.2) which follows from the neutrality assumption. In both cases the smoothing takes place over a distance corresponding to a few Debye lengths. To demonstrate this, let us differentiate the reduced electron momentum equation (2.9) to obtain

$$T_e(\ln n_e)_{xx} = -eE_x + (\nu_{\text{eff}} V_e)_x. \tag{3.4}$$

From the Poisson equation $E_x = \frac{e}{\varepsilon_0}(n_i - n_e)$ we have

$$-\frac{\varepsilon_0}{e^2} T_e(\ln n_e)_{xx} + n_e = n_i - \frac{\varepsilon_0}{e^2}(\nu_{\text{eff}} V_e)_x. \tag{3.5}$$

Now one notices that $\nu_{\text{eff}} V_e$ is a "zero order approximation" for the electric field, or more precisely, for $\mathcal{E} := \frac{\varepsilon_0}{e} E$. This follows strictly from (2.9) after neglecting the pressure term $\frac{1}{n_e}(n_e T_e)_x$, which at an average is one order smaller then $eE$. Therefore $\frac{\varepsilon}{e^2}(\nu_{\text{eff}} V_e)_x$ can be estimated as being of the order of $(n_i - n_e)$ and thus it can be neglected in comparison with $n_i$.

Therefore one may assume

$$-\frac{\varepsilon_0}{e^2} T_e(\ln n_e)_{xx} + n_e = n_i. \tag{3.6}$$

Introducing the Debye length

$$\lambda_D = \left(\frac{\varepsilon_0 T_e}{e^2 \tilde{n}_e}\right)^{1/2},$$

where $\tilde{n}_e$ is the typical (or average) electron density we may expand $\ln(1+\xi) = \xi + \dots$ or

$$(\ln\, n_e)_{xx} = \left(\ln(1 + \frac{n_e - \tilde{n}_e}{\tilde{n}_e})\right)_{xx} \simeq \left(\frac{n_e}{\tilde{n}_e}\right)_{xx} + \dots$$

Thus with a good approximation (3.6) can be replaced by linear Helmholtz equation

$$-\lambda_D^2 (n_e)_{xx} + n_e = n_i, \tag{3.7}$$

which indeed, similarly to (3.6), shows that $n_e$ has (in the limit $\frac{m_e}{m_i} \to 0$ and $\frac{n_i - n_e}{n_e} \to 0$) better differentiability with respect to $x$ than $n_i$.

The solution to (3.7) in the whole $\mathbb{R}^1$ can be written as the convolution

$$n_e(x) = \int_{\mathbb{R}^1} G(x-y)\, n_i(y)\, dy, \tag{3.8}$$

where $G(x-y) = \frac{1}{2\lambda_D} \exp\left(-\frac{|x-y|}{\lambda_D}\right)$. Thus Eq.(3.8) represents the next approximation, beyond the neutrality assumption $n_i = n_e$ and shows how in this approximation the fact that electric field satisfies Poisson equation influences the electron density distribution. When $\lambda_D \to 0$ the kernel $G(x-y)$ tends to the Dirac distribution $\delta(x-y)$ and in the limit $n_e = n_i$. Otherwise the RHS of (3.8) represents the averaging of $n_i$ on distances of a few Debye lengths.

**Assumption 1.** In the following we will postulate this representation of $n_e$ also in the case of a finite domain $[0, L] \subset I\!R^1$ extending $n_i$ on the whole $I\!R^1$ by suitable constant values it takes at the end points of this domain.

Similar procedure can be applied to obtain a better approximation of the electric field, i.e. better than this which is given by Eq.(3.2) and the neutrality assumption $n_i = n_e$. Again expressing $n_e = n_i - \frac{e}{\varepsilon_0} E_x$ from the Poisson equation we obtain for the pressure gradient

$$(n_e T_e)_x = (n_i T_e)_x - \frac{e}{\varepsilon_0} T_e E_{xx}.$$

Applying this relation in Eq.(3.4), one arrives again at the Helmholtz equation for $E$

$$-\lambda_D^2 E_{xx} + E = -\frac{1}{e} \nu_{\text{eff}} V_e - \frac{1}{e n_e} (n_i T_e)_x, \tag{3.9}$$

where this time $\lambda_D$ is the local Debye length. From the expression for the total current, $I = e(n_i V_i - n_e V_e)$, $V_e$ can be computed to obtain

$$\nu_{\text{eff}} V_e = \nu_{\text{eff}} \left( \frac{I}{e n_e} - \frac{n_i}{n_e} V_i \right).$$

Due to relatively small Debye length the averaging (3.8) is influencing essentially only those terms which are not smooth enough, so one can assume

$$E = \frac{1}{e} \nu_{\text{eff}} \left( \frac{I}{e n_e} - V_i \right) - \frac{1}{e n_e} (n_e T_e)_x, \tag{3.10}$$

where $n_e$ is given by (3.8).

Since in the hypersonic approximation the second term on RHS of Eq.(3.10) is neglected, thus we arrive at

## 3.2 Regularized hypersonic approximation

$$\begin{aligned} \frac{\partial}{\partial t} N_a + V_a \frac{\partial N_a}{\partial x} &= -\beta N_a n_e, \\ \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i V_i) &= \beta N_a n_e, \\ \frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} &= \frac{e}{M} E - \beta N_a (V_i - V_a), \end{aligned} \tag{3.11}$$

where according to Eq.(3.8)

$$n_e = G * n_i$$

and

$$E = \frac{1}{e} \nu_{\text{eff}} \left( \frac{I}{e n_e} - V_i \right). \tag{3.12}$$

$I(t)$ can be computed from the boundary condition for the electric potential $U_0$ similarly as in (2.14).

The solutions, if they exist, are defined for $x \in [0, L]$ and $t \geq 0$. To analsye the problem of existence of local solutions we will use, however, the Lagrange coordinates associated with ion velocity, i.e. coordinates $(t, x_0)$ which are for all times attached to the particles. The relation between these two coordinate systems is given by the functions $x = \tilde{x}(t, x_0)$ which, for any constant value $x_0 \in [0, L]$, describes the particle trajectory as a function of time. This trajectory starts from $x = x_0$ for $t = 0$. However, the trajectories are leaving the interval $[0, L]$ at different moments of time, depending on the starting point $x_0$. This fact can create certain formal difficulties. For this reason, we will prolong formally each trajectory after it leaves the domain, by assuming that the "particle" moves with the constant velocity, it had at the moment when it passed $x = L$. In this way all trajectories will be defined for $t \geq 0$, and thus the function $x = \tilde{x}(t, x_0)$ and the velocity $\frac{d\tilde{x}}{dt}$ will be defined for all positive values of $t$. We shall also assume that $N_a$ – the density of neutrals is a given sufficiently regular function $N_a(t, x)$. More general assumption that $N_a$ functionally depends on $n_e$, $V_i$ is also possible which in fact allows one for treating $N_a$ as a solution of the first of Eqs (3.11). As shown, however, in [7] this equation does not describe adequately the real flow of neutrals which appears to be much more complex. It seems therefore to be

reasonable at this stage to assume $N_a$ as a given function of $t$ and $x$. Finally, instead of the ion density $n_i(t,x)$ we will use the total numbers of ions $N(t,x)$ contained in the interval $[0,x]$ at time $t$

$$N(t,x) = \int_0^x n_i(t,x')\,dx'. \tag{3.13}$$

Denoting $q = \beta N_a$ and integrating the second of Eqs (3.11) we arrive at the following system

$$\begin{aligned} \frac{\partial N}{\partial t} + V_i \frac{\partial N}{\partial x} &= V_a n_0 + \int_0^x q\, n_i\, dx', \\ \frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} &= \nu_{\text{eff}} \left( \frac{I}{e n_e} - V_i \right) - q(V_i - V_a), \\ I(t) &= \left( \frac{1}{e^2} \int_0^l \frac{\nu_{\text{eff}}}{n_e}\, dx \right)^{-1} \left\{ U_0 + \frac{1}{e^2} \int_0^l \nu_{\text{eff}}\, V_i\, dx \right\}, \\ n_e &= G * n_i, \end{aligned} \tag{3.14}$$

where in the first equation $n_e$ is replaced by $n_i$ in accordance with assumption of quasi neutrality of plasma. This cannot be done in the equation for $V$, otherwise we would obtain a system which exhibits Hadamard instability. In the Lagrange coordinates we have

$$\begin{aligned} \frac{dN}{dt} &= V_a n_0 + \int_0^x q n_i\, dx', \\ \frac{dV}{dt} &= \nu_{\text{eff}} \left( \frac{I}{e n_e} - V_i \right) - (\beta N_a - \sigma)\,(V_i - V_a), \\ \frac{dx}{dt} &= V_i, \end{aligned} \tag{3.15}$$

which can be considered as an ordinary differential equation in the Banach space $C^{\,1}(O,L)$ of differentiable (or $C^{0,1}$ Lipschitz continuous) functions.

## 3.3 Weak solutions

Following the idea sketched in Introduction we will consider now $V(t,x)$ as a multivalued function. For brevity we will assume that the initial conditions $V_0(x)$ and $N_0(x)$ are Lipschitz continuous functions $V_0,\, N_0 \in C^{0,1}[0,L]$. We will seek solutions in the system of coordinates $(t, x_0)$ which employs characteristics, i.e. in the Lagrangian coordinates.

Let then $x = \tilde{x}(t,x_0)$ satisfies the equation $\frac{dx}{dt} = \tilde{V}(t,x_0),\ \ \tilde{x}(0,x_0) = x_0$. To define the "total" density $n(x)$ which includes all species of particles moving with different velocities (which happens when $\tilde{x}_{\,(t)}(x_0)$ is not invertible) we define, at first, the total number $N(t,x)$ of particles contained between 0 and $x$ at time $t$. Similarly, in accordance with our convention, $\tilde{n}(t,x_0)$ denotes the density of those ions which are moving along the characteristic starting from $x_0$ at $t = 0$. As time elapses the density changes, because new ions are produced at the rate which is proportional to $\tilde{n}(t,x_0)$ and because the distance between the neighbouring characteristics changes. Thus $\tilde{N}(t,x_0)$ will denote the total number of ions moving at time $t$ along all the characteristics which have started at $t = 0$ from all points of the interval $[0, x_0]$.

Having a continuous bounded nondecreasing function $\tilde{N}(x_0)$ – the number of ions between 0 and $x_0$ expressed in the Lagrange coordinates, we will treat it as a cumulative distribution of a positive measure $\tilde{m}$ defined on the Borel subsets of the interval $[0,L]$. Then if $x = \tilde{x}(x_0)$ is the transformation from the Lagrange to the Euler coordinates, we define

$$N(x) = \tilde{m}\ \left\{\, x^{-1}([0,x])\ \right\}. \tag{3.16}$$

It is to say, $N(x)$ is the measure (defined by $\tilde{N}$) of the set of those points whose image by $x(\cdot)$ lies in $[0,x]$. Let us notice that $N(x)$ is a bounded nondecreasing function, hence a $BV$–function.

Now we can consider the system (3.14) in the Lagrange coordinates, where $N(t,x)$ is defined for each $t$ by (3.16). We may note that in the case when $x$ is one to one, the definition (3.16) implies that $\tilde{N}(t,x_0) = N(t,(t,\tilde{x}_0))$.

To define weak solutions for our regularized hypersonic approximation we must still modify the system of equations. First of all for each fixed $t$ we must define the total ion density $n_i(x)$ admitting the fact that the velocity of ions can become multivalued and consequently the total ion density $n_i(x)$ will be composed, in general, of groups of ions moving

with different velocities. We need to define the total ion density in order to compute the electron density which, due to the neutrality condition, should be approximately equal to the ion density, or after applying the corrections following from the "Poisson equation" to

$$n_e(x) = \int_R G(x-y)\, n_i(y)\, dy = \int G(x-y)\, dN(y). \tag{3.17}$$

In principle, our functions are defined on the interval $[0, L]$. To define the above integral we assume that for $x < 0$ we have $N(x) = N(0) = 0$, whereas for $x > L$ we assume that $N(x) = N(L)$.

The electron density is needed in order to determine the electric field from Eq. (3.9). One notices that Eqs (3.10) has no meaning in the case when $V_i(t, x)$ becomes a multivalued function. Therefore we have to go back to the equation for the total current

$$I(t) = I_i(t, x) - en_eV_e, \tag{3.18}$$

where by $I_i$ we denoted the ion current. Although the total current is not dependent on $x$, the ion and electron currents are dependent. We also assume here that $n_e$ and $V_e$ are single valued functions. This, in fact, follows from the construction since $n_e(x)$ is defined by Eq. (3.17) and $V_e$ is defined by equation (3.18). In order to determine the ion current we first define the total ion momentum (divided by ion mass)

$$\tilde{I}_i(t, x_0) = \int_0^{x_0} \tilde{V}(t, x_0')\, d\tilde{m}_{(t)}(x_0'), \tag{3.19}$$

which again is treated as a cumulative distribution of a certain measure. Let us denote this measure by $\tilde{j}$ then the ion current is equal to the density of the measure defined by the BV function

$$J(t, x) = \tilde{j}\, \left\{ \tilde{x}^{-1}_{(t)}([0, x]) \right\}, \tag{3.20}$$

or more precisely

$$I_i(t, x) = e\, \frac{\partial}{\partial x}\, \tilde{j}\, \left\{ \tilde{x}^{-1}_{(t)}([0, x]) \right\}. \tag{3.21}$$

Now, after applying the regularization procedure to the ion current and to the other terms involving $n_i$ we arrive at the following expression for the electric field

$$E = \nu_{\text{eff}}\, \frac{1}{en_e}\, (I - G * I_i) - \frac{1}{en_e}\, (n_eT_e)_{,x}, \tag{3.22}$$

which is a generalization of Eq. (3.10). Clearly in the case of hypersonic approximation the second term on the right is neglected and we have

$$E = \nu_{\text{eff}}\, \frac{1}{en_e}\, (I - G * I_i). \tag{3.23}$$

Similarly as before, we will use the Lagrange coordinates and the functions in the Lagrange coordinates will be denoted by tilde, so we write $\tilde{N}(t, x_0)$, $\tilde{V}(t, x_0)$, $\tilde{x}(t, x_0)$. To formulate our problem in the form of an ordinary differential equation in a Banach space we select the space of bounded Lipschitz continuous functions $C^{0,1}[0, L])$ with appropriate norm .

Thus we will assume that for any fixed $t \in \mathbb{R}_+$ the vector-function

$$(\tilde{N}(t, x_0),\, \tilde{V}(t, x_0),\, \tilde{x}(t, x_0)) \in C^{0,1}[0, L].$$

In addition, motivated by physical reasons we will assume that the ion density $n_i$ is non negative and consequently $\tilde{N}(0, x_0)$ and then $\tilde{N}(t, x_0)$ are nondecreasing functions of $x_0$.

Having a bounded continuous nondecreasing function $\tilde{N}(t, x_0)$ for each fixed $t$ we will treat it as a cumulative distribution of a positive measure $\tilde{m}_{(t)}$ defined on the Borel subsets of the interval $[0, L]$. Let then $x = \tilde{x}(t, x_0)$ be a transformation from the Lagrangian coordinate $x_0$ to the Eulerian $x$, then we define the nondecreasing function of $x$

$$N(t, x) := \tilde{m}_{(t)}\, \left\{ \tilde{x}^{-1}_{(t)}([0, x])) \right\}, \quad 0 \le x \le L. \tag{3.24}$$

It is to say that $N(t, x)$ is the measure of the set of those points $x_0$ which are mapped by $\tilde{x}_{(t)}(\cdot)$ on points laying in the interval $[0, x]$. Clearly, so defined $N(t, x)$ is a nondecreasing bounded function of $x$.

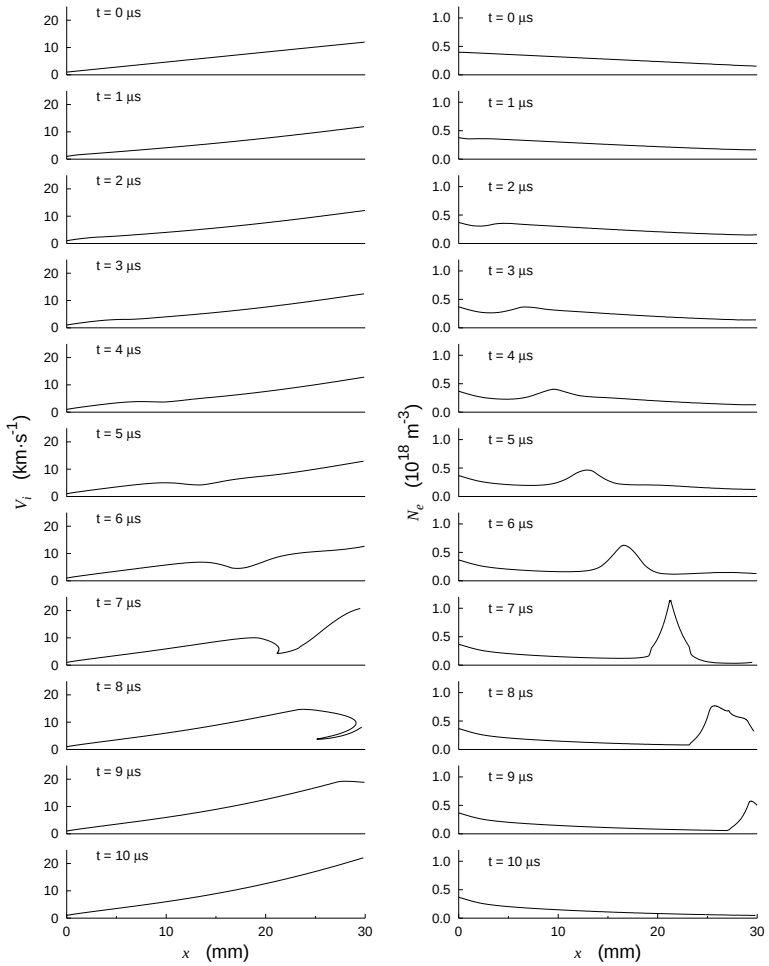


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Figure 1: Example of a solution with shocks, ion velocity (left), electron density (right)

Thus our weak solutions in the case of hypersonic approximation should satisfy

$$\frac{\partial N}{\partial t} + V_i \frac{\partial N}{\partial x} = \int_0^x q\, dN, \tag{3.25}$$

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} = \nu_{\text{eff}} \frac{1}{e n_e} \left(I(t) - G * I_i\right) - q(V_i - V_a)$$

in some generalized sense which we are going to explain. Here $n_e(x)$ and $I_i(t,x)$ are given by (3.17) and (3.21) respectively. One must also slightly modify here the expression for the total current $I(t)$ in (3.14), which in accordance with Eqs (3.19) and (3.20) is

$$I(t) = \left( \int_0^L \frac{\nu_{\text{eff}}}{e n_e}\, dx \right)^{-1} \left\{ U_0 + \int_0^L \frac{\nu_{\text{eff}}}{n_e}\, dJ \right\}, \tag{3.26}$$

and where the integral with respect to $dJ$ ($J$ is defined in (3.20)) is understood as the Lebesgue–Stieltjes integral.

Again the system (3.25) can be written along the characteristics as the equation in Banach space of Lipschitz continuous functions on the interval $[0, L]$

$$\frac{d\tilde{N}}{dt} = \int_0^{x_0} q(\tilde{x}(t, x_0)\, d\tilde{N},$$

$$\frac{d\tilde{V}_i}{dt} = \frac{\nu_{\text{eff}}}{e n_e} \left\{ I(t) - \left(G * I_i\right)(t, \tilde{x}) \right\} - q(\tilde{V}_i - V_a), \tag{3.27}$$

$$\frac{d\tilde{x}}{dt} = \tilde{V}_i(t, x_0),$$

with a Lipschitz continuous initial conditions.

The Fig. 1 shows an example of the numerical solution to the system (3.27) with $I$ assumed to be given and constant, which makes solution more stable. Still, starting from the linear, so very smooth, velocity field we may notice the appareance of shocks, although the solution finally becomes stationary.

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