# Lower-Hybrid Instability in Hall Thruster

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**Abstract:** **This paper is devoted to the development of theoretical model of electron-ion instability in Hall thruster (HT) for the frequency range of the order of lower-hybrid frequency which has a magnitude of $f \approx 1$ MHz. An analysis was carried out within the frames of two-fluid magnetic hydrodynamics. Electron inertia is taken into account in the first order of the ratio of oscillations frequency to electron cyclotron frequency. It is shown that making some additional assumptions it is possible to reduce the MHD equations to one second-order equation with variable coefficients. This equation is similar to Rayleigh's equation but there is a significant difference – additional resonant factor resulting from the disturbed ion motion. The boundary eigenvalue problem for this equation is solved using mainly numerical methods. It is shown that for the electric field profiles in acceleration channel typical for present-day HT and for typical channel dimensions a strong large-scale instability close to aperiodic arises. The "driving force" of this instability is the electron drift velocity nonuniformity. Since both electrons and ions are involved in low-hybrid instability it can cause the enhanced electron transfer in channel. The possible means of an experimental identification of the instability are also considered in the paper.**

## Nomenclature

| | | |
|---|---|---|
| $B$ | = | induction of magnetic field |
| $d$ | = | length of the area filled with the unperturbed electric field |
| $e$ | = | unit positive charge |
| $E$ | = | electric field strength |
| $k$ | = | projection of a wave vector along Y axis |
| $m$ | = | azimuthal wave number |
| $m_e$ | = | electron mass |
| $M$ | = | ion mass |
| $n$ | = | number density of plasma; longitudinal wave number |
| $n_e$ | = | number density of electrons |
| $n_i$ | = | number density of ions |
| $u$ | = | electron velocity |
| $U_d$ | = | discharge voltage |
| $V$ | = | ion velocity |
| $\beta$ | = | parameter defined by (19) |
| $\gamma$ | = | growth rate of instability |
| $\varepsilon_0$ | = | vacuum dielectric constant |
| $\Phi$ | = | perturbation of potential |
| $\omega_e$ | = | electron cyclotron frequency |

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$\omega_i$ = ion cyclotron frequency
$\omega_{LH}$ = lower hybrid frequency
$\omega_{pe}$ = electron Langmuir frequency

## I. Introduction

IN spite of almost four decades of Hall Thruster (HT) research and developments its physics is still not fully understood. This, first of all, is related to instabilities of plasma in acceleration channel of the thruster.

Among these, the two-dimensional electron-ion instabilities having an azimuthal component of electric field are of special interest because they produce an enhanced transfer of electrons in the acceleration channel.
A large wavelength of the electron-ion perturbations, comparable, as a rule, with the scale of nonuniformity of the fields and plasma parameters in acceleration channel, and the necessity of taking into account plasma boundaries significantly complicate the creation of adequate theoretical models of instabilities in HT. Without analyzing in detail the existing theoretical models of electron-ion instabilities in HT, we can note their peculiarity. Most of those models are based on the small-scale perturbation approximation. This is simple and rather universal method. However, an application of this method to the HT not only contradicts the feature of electron-ion perturbations in the acceleration channel, mentioned above but, above all, does not allow revealing the instabilities when these are due to certain structure in the distribution of an unperturbed parameter instead of just a growth or decrease of the parameter along the channel. In this case an application of numerical methods becomes a necessity. This paper is devoted to the studies of plasma stability in HT with respect to an excitation of relatively high-frequency ($k_x V_{x0} << \omega << \omega_e$, where $k_x$ -projection of a wave vector along a direction of ion acceleration, $V_{x0}$ is the unperturbed ion velocity, $\omega_e$ - electron cyclotron frequency) electron-ion perturbations principally taking into account peculiarities of the profile of electric field in the channel. Previously we fulfilled such a program for purely electron perturbations in HT [1].

## II. Theoretical Model of Instability

### A. Basic Assumptions, Equations and Boundary Conditions

The analysis of the stability of the plasma is carried out in the approximation of two-fluid magnetohydrodynamics with cold magnetized electrons and cold non-magnetized ions. We use Cartesian frame with X and Z axes directed along applied electric and magnetic field respectively. The perturbations are assumed to be potential and infinitely spread along the magnetic lines (i.e. they are two-dimensional). Dissipative and ionization processes are neglected. Besides, it is supposed that

1) $$\left|\frac{1}{E_0}\frac{\partial E_0}{\partial x}\right| >> \left|\frac{1}{B_0}\frac{\partial B_0}{\partial x}\right|, \left|\frac{1}{n_0}\frac{\partial n_0}{\partial x}\right|$$

where $E_0, B_0, n_0$, are the unperturbed electric field strength, induction of magnetic field, and number density of the electrons (ions),respectively.

2) The perturbations frequency is high enough that $\omega >> \pi V_{x0}/d$, where $d$ is the length of the area, filled with the unperturbed electric field.

Under these assumptions the linearized MHD equations take the following form

$$\frac{\partial u_x}{\partial t} + u_0 \frac{\partial u_x}{\partial y} = \frac{e}{m_e}\frac{\partial \Phi}{\partial x} - \frac{e}{m_e} u_y B_0$$

$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_{y0}}{\partial x} + u_{y0}\frac{\partial u_y}{\partial y} = \frac{e}{m_e}\frac{\partial \Phi}{\partial y} + \frac{e}{m_e} u_x B_0$$

$$\frac{\partial n_e}{\partial t}+n_0\frac{\partial u_x}{\partial x}+n_0\frac{\partial u_y}{\partial y}+u_{y0}\frac{\partial n}{\partial y}=0 \quad (1)$$

$$\frac{\partial V_x}{\partial t}=-\frac{e}{M}\frac{\partial \Phi}{\partial x}$$

$$\frac{\partial V_y}{\partial t}=-\frac{e}{M}\frac{\partial \Phi}{\partial y}$$

$$\frac{\partial n_i}{\partial t}+n_0\frac{\partial V_x}{\partial x}+n_0\frac{\partial V_y}{\partial y}=0$$

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=\frac{e(n_e-n_i)}{\varepsilon_0};$$

where $u_x, u_y$ -projections of the perturbations of electron velocity on axes X and Y, respectively,
$\Phi$ - perturbation of potential,
$V_x, V_y$ -projections of ion velocity on axes X and Y, respectively,
e-unit positive charge,
$\varepsilon_0$ -dielectric constant of vacuum,
$n_e, n_i$ -perturbations of number densities of electrons and ions, respectively,

$$u_{y0}=-\frac{E_{x0}(x)}{B_0}.$$

The subscript 0 indicates the unperturbed quantities.

We seek the solution of the linearized set of Eqs. (1) in the form:

$$\mathbf{F}(x,y,t)=\mathbf{F}_k(x)\exp(-i(\omega t-ky)) \quad (2)$$

where $\mathbf{F}$ and $\mathbf{F}_k$ are the vector of perturbed parameters and vector of functions dependent on x, respectively, and *k* is the projection of the wave along the Yaxis. In the ring channel *k* can take only the following values:

$k=\frac{m}{r}$ , $m=\pm 1,\pm 2...$-azimuthal wave number.

We substitute the right side of Eq. (2) instead of its left side to Eqs. (1). In the process, we take into consideration the inertia of the electrons only in the first order with respect to $\frac{|\omega|}{\omega_e},\frac{|ku_{y0}|}{\omega_e},\frac{1}{\omega_e}\left|\frac{\partial u_{y0}}{\partial x}\right|$. After reordering and taking into account the fact that in HT the inequality

$$\frac{\omega_{pe}^2}{\omega_c^2} \ll 1 \tag{3}$$

is valid ($\omega_{pe}$-electron Langmuir frequency), Eqs. (1) reduces to the following second order equation with variable coefficients for $\Phi_k$:

$$\frac{d^2\Phi_k}{dx^2} - k^2\Phi_k + \frac{k}{1-\frac{\omega_{LH}^2}{\omega^2}} \frac{\frac{d^2u_{y0}}{dx^2}}{(\omega - ku_{y0})}\Phi_k = 0 \tag{4}$$

Where $\omega_{LH} = \sqrt{\omega_e \omega_i}$ -lower hybrid frequency as inequality (3) is valid,

$\omega_i$ -ion cyclotron frequency.

The obtained Eq. 4 describes lower hybrid perturbations in the plasma with shear of transverse velocity[2]. It differs from Rayleigh's equation, well-known in dynamics of ideal fluid by the presence of the additional resonance factor $\left(1-\frac{\omega_{LH}^2}{\omega^2}\right)^{-1}$.

The following boundary conditions are used for Eq. (4):

$$\Phi_k(0) = \Phi_k(d) = 0 \tag{5}$$

Rayleigh's equation has been used in our previous paper[1] for analysis of the purely electron perturbations. The presence of the additional resonance factor due to motion of ions in the perturbations makes solving the eigenvalue problem (4)-(5) significantly more difficult.

### B. Approximate Analytical Solution for Model Distribution of Drift Velocity

We begin the consideration of the problem (4)-(5) with the model sample. Let the drift velocity distribution in the acceleration channel be defined by

$$u_{y0}(x) = u_{yo\max}\left(\sin^2\frac{\pi x}{d} - \frac{1}{2}\right) \tag{6}$$

Where $u_{y0\max}$ –the maximum value of $u_{y0}$ in the channel.

This drift velocity distribution does not correspond to its real distribution in HT used today because it presupposes the alternating-sign $u_{y0}$ in the channel, for example, due to the variation of the magnetic field direction. But it allows to obtain rather simply the approximate analytical solution of (4)-(5) and to track down in the pure form the influence of the "hybrid" resonant multiplier on the plasma stability in the case that the drift velocity depends on the coordinate.

Substituting the expression for $u_{y0}$ from Eq. (6) to Eq (4) we obtain

$$\frac{d^2\Phi_k}{dx^2} - k^2\Phi_k - \frac{1}{1-\frac{\omega_{LH}^2}{\omega^2}} \frac{k\frac{4\pi^2}{d^2}u_{y0\max}\left(\sin^2\frac{\pi x}{d}-\frac{1}{2}\right)}{\omega - ku_{y0\max}\left(\sin^2\frac{\pi x}{d}-\frac{1}{2}\right)}\Phi_k = 0 \quad (7)$$

At the frequencies $\omega \approx \omega_{LH}$ , as the resonance due to the denominator of the first multiplier in third term of Eq. (4) is significant, the value of the frequency $\omega$ is about an order of magnitude lower than $|ku_{yo\max}|$ . But since $u_{yo}$ depends on the coordinate $x$ according to Eq.6 there always exists $x$ in the interval

$$0 \le x \le d \quad (8)$$

at which the denominator of the third term in Eq.(7) turns to zero ("hydrodynamic" resonance).
However, since

$$|\omega| \sim \omega_{LH} \ll |ku_{y0\max}| \quad (9)$$

this resonance will take place at the vanishing values of $\sin^2\frac{\pi x}{d}-\frac{1}{2}$. But the same term appear as a factor in the numerator of the third term in Eq. (7). It means that nearby those two points in the interval (8) where the denominator vanishes the numerator will also vanish and the contribution of there regions at the integration of Eq. (7) will be insignificant. In the other regions of the interval (8), $|\omega| \ll \left|ku_{yo\max}\left(\sin^2\frac{\pi y}{d}-\frac{1}{2}\right)\right|$. Therefore in the first approximation $\omega$ in the denominator of the second multiplier of the third term can be omitted. Then Eq.(7) is reduced to

$$\frac{d^2\Phi_k}{dx^2} - \left(k^2 - \frac{4\pi^2}{d^2}\right)\Phi_k = 0 \quad (10)$$

This is the equation with constant coefficients. The characteristic equation corresponding to it is

$$s^2 - \left(k^2 - \frac{4\pi^2}{d^2}\right) = 0 \quad (11)$$

The solution of he Eq.(8) has the following form

$$\Phi_k = Ae^{s_1 x} + Be^{s_2 x} \quad (12)$$

where

$$s_{1,2} = \pm\sqrt{k^2 - \frac{4\pi^2}{d^2}} \quad (13)$$

Using the boundary conditions (5) we obtain

$$\sqrt{k^2 - \frac{4\pi^2}{d^2}} \cdot d = i\pi n \qquad n=1,2,3... \tag{14}$$

(Here $n$ -longitudinal number of mode.)
or

$$\left(\frac{\omega}{\omega_{LH}}\right)^2 = \frac{\pi^2 n^2 + k^2 d^2}{k^2 d^2 + \pi^2 (n^2 - 4)} \tag{15}$$

At $n=1$ from the dispersion Eq.(15) it follows that as

$$k^2 d^2 < 3\pi^2 \tag{16}$$

an aperiodic instability appears with a growth rate

$$\gamma = \sqrt{\frac{\pi^2 + k^2 d^2}{3\pi^2 - k^2 d^2}}\,\omega_{LH} \tag{17}$$

If $k^2d^2$ is not very close to $3\pi^2$ the instability growth rate is about $\omega_{LH}$, which corresponds to our assumption (9). In the case that $k^2d^2$ is close to $3\pi^2$ the condition (9) is violated and the result becomes incorrect.

The dependence of the instability growth rate on $|kd|$ is shown on Fig. 1. As $k^2d^2>3\pi^2$ the instability disappears and the stable perturbation with the frequencies shown on the Fig. 1 can exist in plasma. Also presented on Fig. 1 are the results of the numerical solution of the eigenvalue problem (4)-(5) for the drift velocity profile (6) using Eq.(4) (The method used for numerical solution will shortly be described in the next part). As is seen from Fig. 1, the results of the approximate analytic solution at $k^2d^2$ not very close to $3\pi^2$ coincide well with obtained numerical solution.

In the case of the drift velocity profiles approximating its real distribution in HT acceleration channel, it is necessary to take into account both resonant factors in the third term of Eq.(4). In order to do it we use numerical solving the problem (4)-(5).

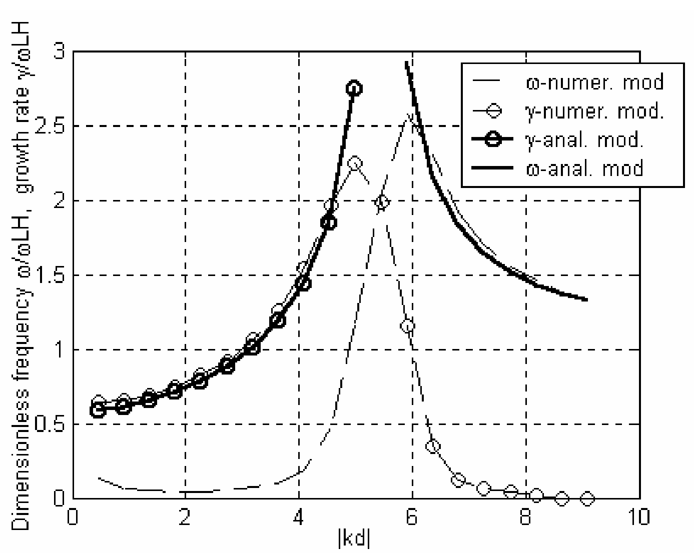


**Figure 1. Dependence of parameters of instability for model sample of drift velocity distribution**

### C. Numerical Solution of Eigenvalue Problem for Profile Approximating Real Drift Velocity Distribution

Let us approximate the real electric field distribution in the region where it exists in HT using the following function (Fig. 2):

$$E_0 = E_{o\max} \sin^2 \frac{\pi x}{d} \qquad (18)$$

This approximation of electric field distribution includes two points of inflection. The results of the experimental investigations of the electric field in HT demonstrate[3-5] that the electric field distribution in the channel has, as a rule, inflection points. This conclusion is also correct for the electron drift velocity calculated from the measured electric and magnetic fields in the acceleration channel.

Boundary eigenvalue problem was solved using the shooting method combined with so-called global-converging Newton-Ralphson iteration procedure.

The calculations were carried out mainly for the dimensions and parameters of the well-known SPT-70 thruster. First of all, it should be noted that the considered instability excites a wave which propagates in the direction of the electron drift that corresponds to the negative values of the mode number m. The wave is localized in the whole area where the applied electric field exists. It can be seen in Fig. 3, where the sample of real and imaginary parts of $\Phi_k$ is presented.

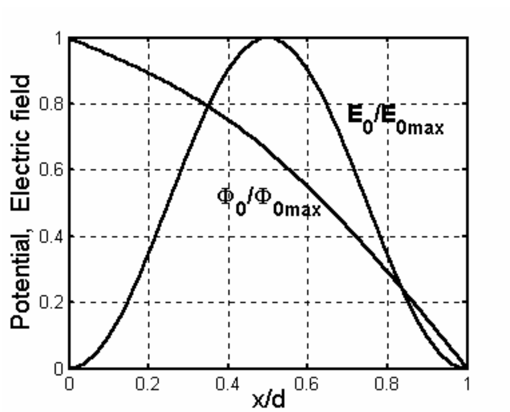


**Figure 2. Approximation of distributions of unperturbed electric field strength and potential in channel**

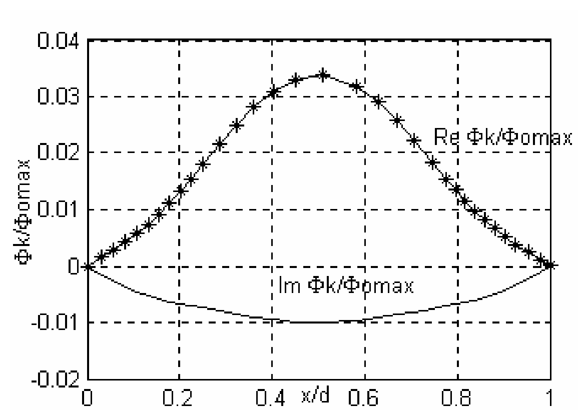


**Figure 3. Sample of localization of wave in acceleration channel**

In Fig. 4 the dependence of the instability growth rate and frequency on the absolute value of mode number *m* for two operation modes corresponding to the following values of parameter $\beta$: 5.53 and 7.7,

where $$\beta = \frac{u_{o\max}}{\omega_{LH} d} \qquad (19)$$

If to assume that $B_0$=0.018 T, then the value of $\beta$ =5.53 corresponds to discharge voltage $U_d$=200 V, and the value of $\beta$ =7.7 corresponds to $U_d$=285 V.

From Fig 4 it is seen that:

1) Instability growth rate for the *m* with small absolute values (*m*=-1 and *m*=-2) exceeds or approximately equals the circular frequency. The instability is close to aperiodic. The values of $\gamma$ and $\omega_r$ are close to low hybrid frequency $\omega_{LH}$.
2) As $|m|$ further increases the growth rate first slightly grows reaching the maximum at *m*=-4 and then rapidly drops to zero at *m*=-6. The oscillation frequency steadily grows as the mode number increases. Its growth is faster then linear.

In order to additionally to clarify the influence of lower hybrid resonance on exciting the oscillations in HT, in Fig. 4, the dependence of the growth rate and frequency on |*m*| is shown by dash lines for the case that the first resonance multiplier in the third term of Eq. (4) is equal to 1. This means that we artificially increased the mass of ion until infinite magnitude. In this case, the frequency and especially the growth rate strongly drop at *m* =-1 and *m* =-2, but after that, they progressively approach the values, obtained by solving the full Eq. (4). The influence of the finite mass of ions on the instability is most pronounced at the small values of *k* (See Fig. 5) that corresponds to HT with large diameter of the channel (thrusters type of SPT-100, SPT-140). In this case the ratio of the growth rate due to perturbed motion of xenon ions to the growth rate at the infinite heavy ions can exceed the order of magnitude. From this consideration it follows that lower-hybrid instability in HT as *k* increases gradually transforms into electron istability[1], described by Rayleigh's equation.

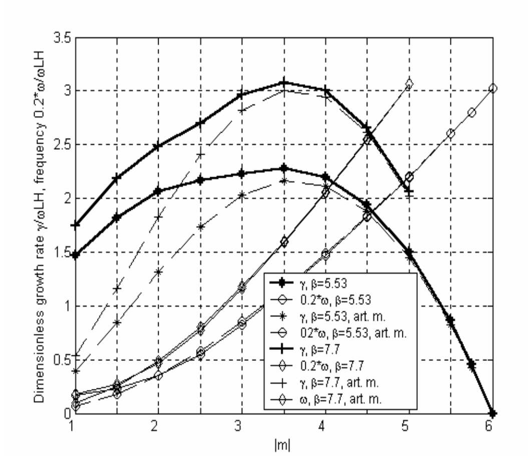


**Figure 4. Parameters of instability as a function of azimuthal mode number**

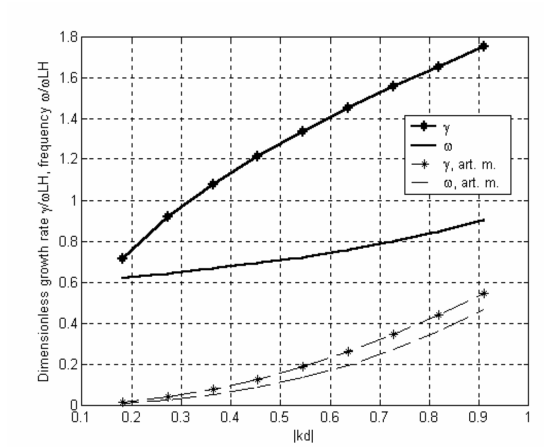


**Figure 5. Parameters of instability at small values of |kd|**

The "driving force" of the considered instability is the nonuniformity of the electron drift velocity. It is immediately seen from Eq. (4). If the second derivative of the drift velocity is zero over the range $0 \le x/d \le 1$, the instability vanishes.

As is well known, according to the Rayleigh's theorem, in the case of boundary eigenvalue problem (4)-(5) but without the factor $\left(1-\frac{\omega_{LH}^2}{\omega^2}\right)^{-1}$ in Eq. (4), the necessary condition of perturbation instability is presence of at least one point of inflection on the $u_{y0}$ profile in the range under consideration. In the case of lower hybrid instability the condition of inflection point presence on the drift velocity profile is also of importance. At least for $u_{y0}$ profiles which not only do not include the inflection point but are also monotonous, the instability was not found at numerical solving the problem (4)-(5).

The fact that near lower hybrid frequency the perturbation frequency growth with the growth of |*m*| is faster than linear points out that these perturbations can decay into two lower-frequency perturbations as a result of so-called nonlinear three-wave process. This should ultimately lead to the formation of turbulent oscillation spectrum.

Taking into consideration the availability of the finite gradient of plasma density and magnetic field induction carried out with numerical solution of the eigenvalue problem leads to some reduction in the growth rate of the instability but does not suppress it. This issue will be considered in detail in another paper.

It should be noted that the potential approximation applied in present work is used near the limit of the domain of its applicability what should be taken into account in further work.

## III. Possibility of Experimental Identification of Lower-Hybrid Instability

As follows from the theoretical model the lower hybrid frequency is strong and it should rise above the background spectrum. From this point of view, the diffuse peak in the region of 1 MHz observed in Ref. 6

seemingly belongs to the oscillations caused be the lower hybrid instability. Besides the expected peak of the oscillation amplitudes in the frequency range of $f$≈0.5-1.5 MHz there is additional important factor that can help in identification of the lower hybrid frequency by relatively simple means. In contrast to several other instabilities whose frequencies reduce with the magnetic induction growth, the frequency of the lower-hybrid instability not only does not reduce, but slightly grows as the $B_0$ increases. For example, as the magnetic induction is increased on 47% the lower hybrid frequency increases on 12%. The identification mentioned above can be accomplished using one near-wall electrical probe. If there is a possibility to place several near-wall electric probes separated in azimuthal and longitudinal directions, the reliability of the instability identification may be increased using the estimation of the wave spatial structure. Perturbations caused by the lower-hybrid instability should propagate in the direction of the electron drift, at the frequencies about 1 MHz should be mode $m$=-1 or $m$=-2 depending on the thruster operation mode and according to Fig, 3 should be localized in the whole region where applied electric field exists.

## IV. Conclusion

1. The theoretical model of electron-ion plasma instability in Hall thruster in the region of lower-hybrid frequency has been developed.
2. The “driving force” of the instability is the electron drift velocity nonuniformity.
3. The instability is localized in the whole region where the applied electric field exists.
4. It is shown that the instability is strong and for the parameters typical for present-day HT models is close to aperiodic.
5. In the experiments the instability should be observed as the distinct peak in the area of the spectrum $f$≈0.5-1.5 MHz. Additional features that allow the reliable identification of the lower-hybrid frequency are:
   a) growth of the frequency with the increase of magnetic field induction;
   b) large wavelength, comparable with the azimuthal size of the acceleration channel;
   c) the propagation of the wave in the direction of the electron drift;
   d) localization of the wave in the whole region with strong electric field.

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