A High Order Relativistic Particle Push Method for PIC Simulations

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Martin Quandt* and Claus-Dieter Munz†
Institut für Aerodynamik und Gasdynamik,
Universität Stuttgart, Germany

Rudolf Schneider‡
Institut für Hochleistungsimpuls und Mikrowellentechnik,
Forschungszentrum Karlsruhe, Germany

Abstract: Within the particle treatment module in a PIC code the particle push algorithm computes the changes in the velocity and position of a particle. For these calculations with an effective order of convergence up to 2 only the current values of the electromagnetic (em) fields at the position of the particle are needed. The recent developement of high order field solver enables the additional use of higher derivates in time of the em field values to calculate the particle motion with an increased accuracy. We propose a high order relativistic particle push method which has the ability to compute the new particle values up to the order of the used field solver. The desired order of this particle method depends only on the availability of the derivates in time of the em field and can be linked with different kind of field solvers like finite difference, finite element or finite volume schemes. Based on the relativistic Lorentz equation of motion for charged particles in any em fields, we transform the equation in a way which makes it possible to get an approximately solution of desired accuracy with a taylor series expansion in time. The evaluation of all terms at one stage is not possible because of the unknown higher derivates of the Lorentz factor. With the given values of the em field the first term can be calculated and constructive on this the next derivate of the Lorentz factor. The general principle of successive evaluating can repeated as often as desired provided that all needed derivates of the em field are given. The analysis of the obtained accuracy and the determination of the effective order of convergence is performed with 3 examples. The analytic solution of the first non relativistic example is known as the Lissajou figures. In this case the magnetic field is set to zero and the particle motion is only determined by the applied electric field. The second relativistic case shows the motion of a particle only in a magnetic field and the last example of the relativistic ExB drift closures the study of convergence here.

*Scientist, quandt@iag.uni-stuttgart.de
†Professor, munz@iag.uni-stuttgart.de
‡Senior Fellow, rudolf.schneider@ihm.fzk.de

Nomenclature

$\vec{E}$	$[V/m]$	= electric field
$\vec{B}$	$[Vs/m^2]$	= magnetic field
$\vec{U}$	$[m/s]$	= relativistic velocity
c	$[m/s]$	= speed of light
m_0	$[kg]$	= mass of particle
q	$[As]$	= particle charge
t	$[s]$	= time
$\vec{v}$	$[m/s]$	= velocity of the particle
$\vec{s}$	$[m]$	= position of the particle
γ,$\gamma^\star$	$[-]$	= Lorentzfactor
EOC	$[-]$	= effectiv order of convergence
PIC	$[-]$	= Particle In Cell
eN	$[-]$	= euclidian norm

I. Introduction

The recent development of high order field solver[12,16,17] for the computation of electromagnetic wave propagation causes the possibility to solve the relativistic Lorentz equation with a distinctive increase in accuracy and efficiency. To get the high order of the solution the field solver have to calculate in most cases the higher derivates in time of all variables. In consequence of the interaction of the particle treatment module with the field solver it is necessary to improve the accuracy of both modules to keep on an efficient entire PIC cycle. The errors which accumulate over the time[10] in position and velocity of the particle may influence the field solution via the source terms of the field equation. In some cases it is possible to reduce these errors only with an increase in the steps of calculating the particle quantities. In simulations with a negligible number of particles it is a suitable strategy. A more elegant alternative is the use of methods which are capable to participate on the improved abilities of high order field solver. In figure (1) is schematically shown a part of a PIC cycle where the particle treatment module is located after the field solver. Further

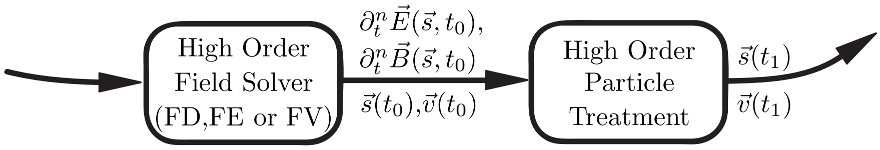

Figure 1. Part of PIC a cycle where the particle treatment modul calculates the new position and velocity of the particle with the available higher derivates in time of the em-field computed by the high order field solver.

extension of the particle treatment like a DSMC[19] or FokkerPlanck[18] module may be located here in parallel to the Lorentz particle push as realized in the PicLas Code[14,15] or in a serial manner. For our proposed method all needed derivates in time of the em field at the position of the particle is a prerequisite and can provided of most of the recent developed field solvers. With these information of the em-field and a successive evaluation of the terms in the taylor series expansion it is possible to get also the higher derivatives of the Lorentz factor γ. The additional field solver request can fulfilled without general design modification. Only the number of points of evaluating the function for the em field distribution is expanded with the particle positions. To present our method we start in the next part with the numeric approximation of the Lorentz equation. For the taylor series expansion in time a suitable redraft is made. With this approximation the relativistic motion of a particle can solved up to the desired accuracy. With the availability of all of the em field derivates the Lorentz factor γ and its derivates can be evaluated. In the first example the non relativistic motion of a particle is considered only in the applied time varying E-field. The solution is known as the Lissajou figures. The second case is the relativistic motion only in a B-field. A more complex situation is the last example and known as the $E \times B$ drift.

II. Equation of Motion for Charged Particles

We limit our focus and consider only the particle motion in applied em field which can be described by the relativistic Lorentz equation. For the derivation to the used form we refer to literature.[1] Starting from this point the relation of the velocity and the em Field can be written as a system of differential equations. With the abbreviation of the operator $\frac{\partial}{\partial t} = \partial_t^1$ we can write the Lorentz equation as follows:

$$\partial_t^1 m_0 \gamma(t) \vec{v}(t) \quad = \quad q \left[\vec{E}(\vec{s}, t) + \vec{v}(t) \times \vec{B}(\vec{s}, t) \right] \tag{1}$$

The mass of a particle is noted with m_0 with the charge q and the velocity $\vec{v}$ at time t with the position of $\vec{s}$. The transformation of the velocity is given by $\vec{U}(t) = \gamma(t)\vec{v}(t)$ whereas the Lorentz factor can be calculated with $\gamma(t) = 1/\sqrt{1 - \vec{\beta}(t)^2}$ and with the relation of the vector $\vec{\beta}(t) = \vec{v}(t)/c$. The constant speed of light in vacuum is denoted with c. The quantities of the electric field is stated with $\vec{E}$ and with $\vec{B}$ the magnetic part.

A. Numerical Approximation of the Lorentz-Equation

To evaluate the taylor series expansion in time of eq. (1) it is useful to substitute the velocity on the right hand by the relation of $\vec{v} = \gamma^\star(t)\vec{U}(t)$. The star indicates the use of the relativistic velocity $\vec{U}$ in the calculation of $\gamma^\star(t) = 1/\sqrt{1 + \vec{\beta}^\star(t)^2}$ and for the vector $\vec{\beta}^\star(t) = \vec{U}(t)/c$. With these simplification eq.(1) can written in the following form:

$$\dot{\vec{U}}(t) \quad = \quad \vec{\mathcal{E}}(\vec{x}, t) + \gamma^\star(t)\vec{U}(t) \times \vec{\mathcal{B}}(\vec{x}, t) \tag{2}$$

For the sake of simplicity we collect q/m_0 and introduce the fields $\vec{\mathcal{E}}(\vec{x}, t) = q/m_0\vec{E}(\vec{x}, t)$ and $\vec{\mathcal{B}}(\vec{x}, t) = q/m_0\vec{B}(\vec{x}, t)$. On eq. (2) we can apply the taylor series expansion in time which can evaluated up to the desired accuracy and is only limited by the given derivates of the $\vec{E}$ and $\vec{B}$ fields. The general rule to evaluate the terms of the taylor series includes the Leibnitz rule for a product. We made additional abbreviation for a plainly formulation $\dot{\vec{U}}(t_0) = \dot{\vec{U}}_0$, $\vec{\mathcal{E}}(\vec{x}, t_0) = \vec{\mathcal{E}}_0$, $\vec{\mathcal{B}}(\vec{x}, t_0) = \vec{\mathcal{B}}_0$ and $\gamma^\star(t_0) = \gamma_0^\star$:

$$\left[\dot{\vec{U}}\right]_{t_0}^{(\nu)} = \partial_t^n \dot{\vec{U}}_0 \quad = \quad \partial_t^\nu \vec{\mathcal{E}}_0 + \sum_{\nu=0}^{n} \frac{n!}{\nu!(n-\nu)!} \left(\partial_t^{n-\nu}\gamma_0^\star\right) \left[\vec{U}_0 \times \vec{\mathcal{B}}_0\right]^{(\nu)} \tag{3}$$

The parameter n indicates the level of the time derivative of the variable $\dot{\vec{U}}$ and the upper bound and parameter of the sum are noted with ν, k. According to the Leibnitz rule the crossproduct $\vec{E} \times \vec{B}$ can be written as follows:

$$\left[\vec{U}_0 \times \vec{\mathcal{B}}_0\right]^{(\nu)} = \partial_t^\nu \left(\vec{U}_0 \times \vec{\mathcal{B}}_0\right) \quad = \quad \sum_{k=0}^{\nu} \frac{\nu!}{k!(\nu-k)!} \left[\vec{U}_0\right]^{(\nu-k)} \times \left[\vec{\mathcal{B}}_0\right]^{(k)} \tag{4}$$

$$\left[\vec{U}_0\right]^{(\nu-k)} \quad = \quad \left(\partial_t^{\nu-k}\vec{U}_0\right) \tag{5}$$

$$\left[\vec{\mathcal{B}}_0\right]^{(k)} \quad = \quad \left(\partial_t^k \vec{\mathcal{B}}_0\right) \tag{6}$$

To calculate the high order terms of the taylor series expansion the lower terms have to be calulated at first. At the time t_0 the first derivate $\dot{\vec{U}}(t_0)$ is given by eq. (2). Using this value the first derivative of the Lorentz factor $\gamma^\star$ can be computed. When the information of the derivative in time of the em field is given then the next derivative of $\dot{\vec{U}}(t_0)$ can be computed and so on. To perform this principle of successive evaluation of the taylor terms all needed derivatives have to be provided by the field solver. For the second order scheme the taylor terms is written with the known first derivative $\dot{\vec{U}}_0$:

$$\partial_t^1 \gamma_0^\star = -\frac{\vec{U}_0 \dot{\vec{U}}_0}{c^2 \sqrt{\left(1 + \beta_0^{\star 2}\right)^3}} \tag{7}$$

$$\left[\vec{U}_0 \times \vec{\mathcal{B}}_0\right]^{(0)} = \vec{U}_0 \times \vec{\mathcal{B}}_0 \quad (8)$$

$$\left[\vec{U}_0 \times \vec{\mathcal{B}}_0\right]^{(1)} = \dot{\vec{U}}_0 \times \vec{\mathcal{B}}_0 + \vec{U}_0 \times \dot{\vec{\mathcal{B}}}_0 \quad (9)$$

$$\left[\dot{\vec{U}}_0\right]^{(1)} = \dot{\vec{\mathcal{E}}}_0 + \partial_t^1 \gamma_0^\star \left[\vec{U}_0 \times \vec{\mathcal{B}}_0\right]^{(0)} + \partial_t^0 \gamma_0^\star \left[\vec{U}_0 \times \vec{\mathcal{B}}_0\right]^{(1)} \quad (10)$$

B. Analytic Solution of the Nonrelativistic Lissajou Figures

To examine the effective order of convergence we compare the numeric results with the known analytic solution. In the first example a single charged particle is regarded. The em field consist only in components of the $\vec{E}$ field and the magnetic part is set to 0. The non relativistic solution of the motion is known as the Lissajou figures and can easily computed with 2 integration of the given $\vec{E}$ field distribution. In our case we use sine function $Asin(\omega t) + \phi$ depend on time. With this function the $\vec{E}$ field is constant in space but in time all derivatives exist. With these assumptions and in case of $\gamma(t) = 1$ and therefore $\vec{v}(\vec{x}, t) = \vec{U}(\vec{x}, t)$ with $\vec{B}(\vec{x}, t) = (0, 0, 0)^T$ eq.(1) can simplified to:

$$\partial_t^1 \vec{v}(t) = \frac{q}{m_0} \vec{E}(\vec{x}, t) \quad (11)$$

The analytic solution of the velocity $\vec{v}(t)$ and the position of the particle $\vec{s}(t)$ is in our case the integration in time.

C. Analytic Solution of the Relativistic B-Field Motion

In the second case we switched off the $\vec{E}$ field and tested the method only with the $\vec{B}$ field component. In this relativistic case we take advantage of the constant energy of the particle. Due to this fact the Lorentz factor $\gamma(t)$ and $\gamma^\star(t)$ are constant and can be calculated with the initial value of velocity $\vec{v}(t_0) = (v_{x0}, v_{y0}, 0)^T$. With the given initial value of the position of the particle the differential equation (1) with $\vec{E}(\vec{x}, t) = \vec{0}$ can simplified to eq. (12). The given solution eq. (13) is the real part of the complex result and can be written in a matrix form as follows.

$$\dot{\vec{U}}(t) = \frac{q}{m_0} \gamma^\star(t) \vec{U}(t) \times \vec{B}(\vec{x}, t) \quad (12)$$

$$\vec{U}(t) = \begin{pmatrix} cos(\gamma_0^\star \omega t) & sin(\gamma_0^\star \omega t) & 0 \\ -sin(\gamma_0^\star \omega t) & cos(\gamma_0^\star \omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{x0}/\gamma_0^\star \\ v_{y0}/\gamma_0^\star \\ 0 \end{pmatrix} \quad (13)$$

$$\vec{s}(t) = \frac{1}{\gamma_0^\star \omega} \begin{pmatrix} sin(\gamma_0^\star \omega t) & -cos(\gamma_0^\star \omega t) & 0 \\ cos(\gamma_0^\star \omega t) & sin(\gamma_0^\star \omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{x0} \\ v_{y0} \\ 0 \end{pmatrix} + \begin{pmatrix} s_{x0} \\ s_{y0} \\ s_{z0} \end{pmatrix} \quad (14)$$

The gyration frequency is stated with $\omega = qB_3/m_0$ and $f = \omega/(2\pi)$. With an easy check the solution can be verified. The derivative of eq.(13) should be the same of the evaluation of eq. (12) with eq.(13). In the non relativistic case with $\gamma_0^\star = 1$ the proposed simple check should lead to the same result.

D. Analytic Solution of the Relativistic $\vec{E} \times \vec{B}$ Drift

In the last relativistic example we chose a combination of given $\vec{E}$ and $\vec{B}$ field so that an analytic solution can be calculated as described in detail and full explained in literature.[1] In this example we chose the case of $|\vec{E}| < c|\vec{B}|$ where the classic motion of the particle can observed. To get the analytic solution it is advisable to perform a transformation of the em field into the K' system which has the constant velocity

$\xi = 1/B^2(\vec{E} \times \vec{B})$ towards the initial system K of the given fields. In the K' system the given em-fields are reduced to $\vec{E}' = 0$ and the magnetic field $\vec{B}'$. In the system K' is now the same situation as described in examples 2 where the eq.(1) could simplified to eq.(12) and where the solution was already given. Further attention is needed with the transformation of the fields and for the solution of the velocity and position in the K' to K-system. The rules of field transformation and of $\vec{v}$, $\vec{s}$ between the system K and K' can be written as

$$\vec{E}' = \gamma \left[\vec{E} + c\left(\vec{\beta} \times \vec{B}\right)\right] - \frac{\gamma^2}{\gamma + 1}\vec{\beta}\left(\vec{\beta}\vec{E}\right) \tag{15}$$

$$\vec{B}' = \gamma \left[\vec{E} - \frac{1}{c}\left(\vec{\beta} \times \vec{E}\right)\right] - \frac{\gamma^2}{\gamma + 1}\vec{\beta}\left(\vec{\beta}\vec{B}\right) \tag{16}$$

$$A_0' = \gamma\left(A_0 - \vec{\beta}\vec{A}\right) \tag{17}$$

$$\vec{A}' = \vec{A} + \frac{\gamma - 1}{\beta^2}\left(\vec{\beta}\vec{A}\right)\vec{\beta} - \gamma\vec{\beta}A_0 \tag{18}$$

The inverse transformation formulas can recived by changing the sign of β and with an exchange of the quantities marked with $'$ with its opposite ones. Here is $A_0' = ct'$ the product of the speed of light with the time t' of the system K' and $\vec{A}' = (x', y', z')^T$ the position of the particle. The transformation of the velocity can also performed with these formulas when the time t is exchanged with γ. A more elegant way of transformation with a matrix is also detailed described by.[1]

III. Results

To determine the efficiency and accuracy of our proposed method we perform for all examples the same procedure of varying the points and the order. For a given set of points the deviation in velocity and position of the particle to the known analytic solution is calculated with the final results of the numeric simulation. The effective order of convergence corresponds to the slope of the graph for the error plotted over the points with double logarithmical scale. The deviation of the vectors $\vec{v}$ and $\vec{s}$ is calculated with the Euclidian norm according to eqs.(19,20). All values for EOC in the charts are calculated with eq.(21).

$$eN(\vec{v}) = ||\vec{v}_{num} - \vec{v}_{exact}||_2 \tag{19}$$

$$eN(\vec{s}) = ||\vec{s}_{num} - \vec{s}_{exact}||_2 \tag{20}$$

$$EOC = -\frac{log(\frac{eN_1}{eN_0})}{log(\frac{n_0}{n_1})} \tag{21}$$

A. Numeric Simulation of the Nonrelativistic Lissajou Figures

In example 1 we used a particle with a charge $q = 1$ and a mass $m = 1$ for further simplification so we can recast the Lorentz equation to eq.(19) where all E-field components are sine function with different parameters.

$$\dot{\vec{U}}(t) = \begin{pmatrix} E_{x0} sin(\omega_x t) + \phi_x \\ E_{y0} sin(\omega_y t) + \phi_y \\ E_{z0} sin(\omega_z t) + \phi_z \end{pmatrix} \quad (22)$$

The parameters we used for the amplitudes E_{x0}, E_{y0} and E_{z0} were set for components to 1. The values we use for the angular rate are $\omega_x = 2\pi$, $\omega_y = 2/3\pi$ and $\omega_z = 3/2\pi$. With a phase shift of ϕ_x, ϕ_y and ϕ_z in all cases of $1/2\pi$ the known figures can be observed. In the figure (4) the numeric simulation calculated with an EOC of 5 marked with dots and compared to the analytic solution depicted by a line. After 10 Periods with a total number of points of 160 the deviation to the analytic solution is very small. The evolution of the deviation to the analytic solution for different EOC's and with an increased resolution of points is depicted in figure (3) The change the resolution the number of points for the next level was doubled. With a sufficient number of points all expected theoretical orders could reached in all cases by the simulation.

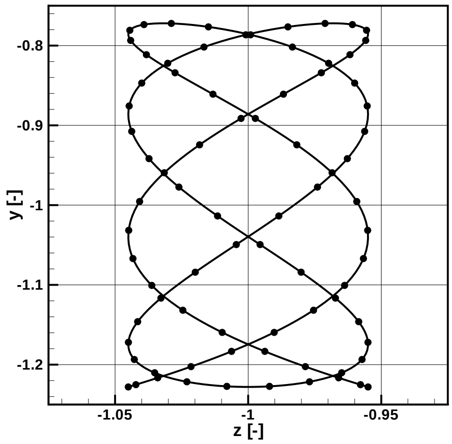

Figure 2. The 3D non relativistic analytic particle motion (line) and the numeric solution (dots) calculated with an EOC of 5 for 10 periods with 160 points.

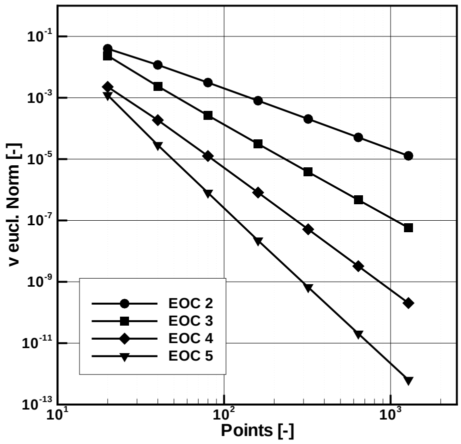

Figure 3. The evolution of the Euclidian error in $\vec{v}$ is plotted for different number of points. The slope of the lines corresponds to the negative EOC.

Table 1. Example 1 with EOC 2

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	3.125e-03		4.118e-03	
80	8.020e-04	1.91	7.718e-04	2.42
160	2.020e-04	1.98	1.667e-04	2.21
320	5.106e-05	1.99	3.896e-05	2.10
640	1.280e-05	2.00	9.439e-06	2.05

Table 2. Example 1 with EOC 3

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	2.650e-04		2.443e-03	
80	3.126e-05	3.16	3.007e-04	3.02
160	3.793e-06	3.04	3.729e-05	3.01
320	4.671e-07	3.02	4.642e-06	3.01
640	5.795e-08	3.01	5.790e-07	3.00

Table 3. Example 1 with EOC 4

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	1.261e-05		2.611e-05	
80	8.126e-07	3.96	9.059e-07	4.85
160	5.148e-08	3.98	3.679e-08	4.62
320	3.238e-09	3.99	1.788e-09	4.36
640	2.030e-10	4.00	9.908e-11	4.17

Table 4. Example 1 with EOC 5

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	7.991e-07		1.155e-05	
80	2.220e-08	5.15	3.513e-07	5.04
160	6.660e-10	5.06	1.082e-08	5.02
320	2.042e-11	5.03	3.356e-10	5.01
640	6.326e-13	5.01	1.046e-11	5.00

B. Numeric Simulation of the Relativistic B-Field Motion

In example 2 we used for the physical quantities in [SI] units the following parameters $q = 1.60218^{-19}C$, $m_0 = 9.10938^{-31}kg$, $c = 2.99792^8 m/s$, $B_z = 0.1Vs/m^2$ and $\vec{E} = (0,0,0)^T$. With the initial velocity in x-direction of $0.6c$ the Lorentz factor $\gamma = 1.25$. The time for the particle motion is set to $10T_{gyro}$ where $T_{gyro} = 2\pi/\omega\gamma^\star$. The motion of the particle has the initial value of $\vec{s}_0 = (0,0,0)^T$. In the numeric simulation

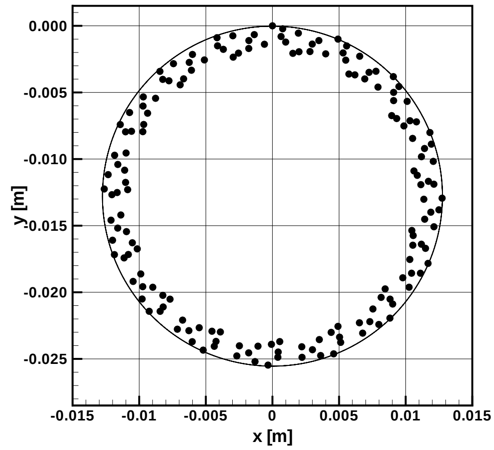

Figure 4. The deviation of numeric solution (dots) with an EOC 3 for the particle position compared to the analytic solution (line) after 10 periods with a total points resolution of 160.

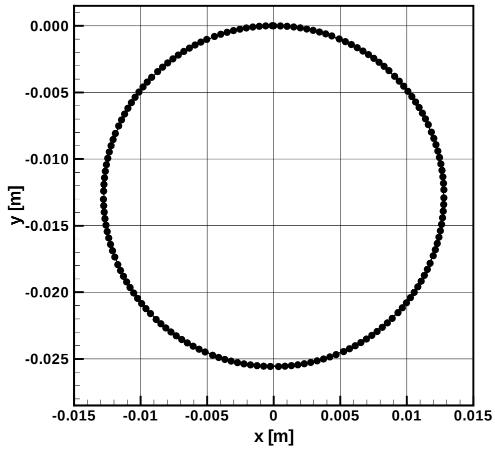

Figure 5. Deviation of the particle position after 10 periods to the analytic circle with the Landau radius is plainly reduced with the same resolution of points.

the calculated values for the velocity and position of the particle are used in the next iteration. For that reason the errors accumulates in time and it is shown in figure (4). In this simulation a total point resolution of 160 is used with an EOC of 3 for the method. In comparison with an EOC of 5 the difference are plainly. The effective order of convergence was calculated for velocity and the position of the particle and is listed in charts below. In all cases the expected order of the method according to the theory could reached with a sufficient number of points. In this relativistic example the Lorentz factor γ is constant and no derivates in

Table 5. Example 2 with EOC 2

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	8.333e+1		1.984e-0	
80	2.430e+1	1.78	5.260e-1	1.92
160	6.345e+0	1.94	5.260e-1	1.96
320	1.613e+0	1.98	3.439e-2	1.98
640	4.061e-1	1.99	8.654e-3	1.99

Table 6. Example 2 with EOC 3

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	2.104e+1		5.121e-1	
80	2.099e+0	3.33	5.318e-2	3.27
160	2.324e-1	3.18	5.998e-3	3.15
320	2.728e-2	3.09	7.107e-4	3.08
640	3.303e-3	3.05	8.644e-5	3.04

Table 7. Example 2 with EOC 4

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	1.144e+0		2.726e-2	
80	7.345e-2	3.96	1.593e-3	4.10
160	4.590e-3	4.00	9.815e-5	4.02
320	2.857e-4	4.01	6.092e-6	4.01
640	1.780e-5	4.00	3.793e-7	4.00

Table 8. Example 2 with EOC 5

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	1.256e+0		2.784e-2	
80	1.537e-1	3.03	3.295e-3	3.08
160	5.517e-3	4.80	1.178e-4	4.81
320	1.753e-4	4.98	3.744e-6	4.98
640	5.464e-6	5.00	1.167e-7	5.00

time exist. A simulation without regarding the relativistic effect may lead already at this point to errors in the final position of the particle.

C. Numeric Simulation of the Relativistic $\vec{E} \times \vec{B}$ Drift

In the last example of a relativistic $\vec{E} \times \vec{B}$ drift we use the same parameters of examples 2 but made only changes in the E-field and set the value to $E_x = 1.0^7 V/m$. We also change the initial value of the velocity from $v_{x0} = 0.6c$ to $v_{x0} = 0.99c$ which correspond to a Lorentz factor of about $\gamma = 7.1$. As the final time we chose the Lorentz invariant Eigen time of the particle. The transformation of the velocity is performed in the same manner as for the position of the particle. In the following figures (6,7), the effect of an EOC of 5 to 3 for the same total number of points of 160 after 10 periods shows the increased accuracy. The positive charge in a positive em field with the given initial velocity starts in both cases at the position $s_0 = (0, 0, 0)^T$. The motion of the particle is characterized by a clock wise rotation with a drift velocity in the negative y-direction. All results of the convergence study with this example are listed below. In the case of EOC 4 the number of points was not enough in this example to reach finally the expected theoretical order. The reduction of the deviation in the velocity and position was greater than it would be necessary for the rate of increased number of points. In all other cases the error diminished finally with the expected order.

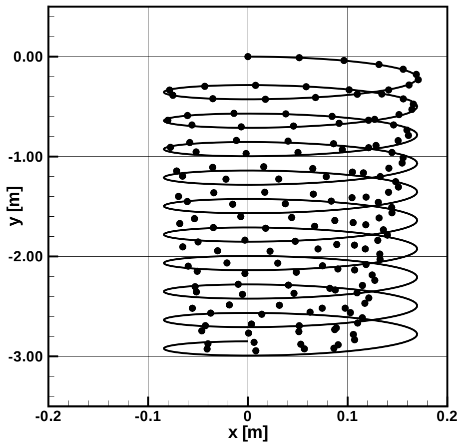

Figure 6. Analytic solution of a ExB drift (line) with the numeric calculation (dots) with an EOC of 3 after 10 Periods

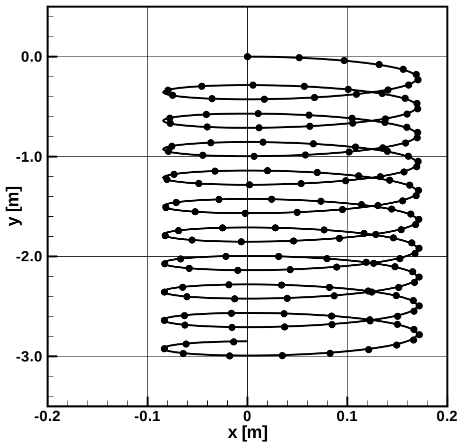

Figure 7. Increased accuracy with the same number of points of the numeric simulation (dots) with an EOC 5 compared to the analytic solution (line).

Table 9. Example 3 with EOC 2

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	1.863e+2		3.149e+1	
80	1.325e+2	0.49	1.674e+1	0.91
160	4.384e+1	1.60	5.362e+0	1.64
320	1.228e+1	1.84	1.488e+0	1.85
640	3.235e+0	1.92	3.902e-1	1.93

Table 10. Example 3 with EOC 3

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	2.000e+2		2.299e+1	
80	2.537e+1	2.98	2.910e+0	2.98
160	3.170e+0	3.00	3.634e-1	3.00
320	3.953e-1	3.00	4.532e-2	3.00
640	4.930e-2	3.00	5.656e-3	3.00

Table 11. Example 3 with EOC 4

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	4.025e+0		4.595e-1	
80	1.458e-1	4.79	1.671e-2	4.78
160	5.658e-3	4.69	6.528e-4	4.68
320	2.449e-4	4.53	2.852e-5	4.52
640	1.198e-5	4.35	1.411e-6	4.34

Table 12. Example 3 with EOC 5

Points	$eN(\vec{v})$	EOC	$eN(\vec{s})$	EOC
40	1.417e+0		2.348e-1	
80	1.260e-1	3.49	1.476e-2	3.99
160	7.312e-3	4.11	8.721e-4	4.08
320	2.619e-4	4.80	3.123e-5	4.80
640	8.467e-6	4.95	1.009e-6	4.95

IV. Conclusion

For the relativistic motion of charged particle in applied em fields described by the Lorentz equation we could present a method which is capable to solve the equation with a desired accuracy. The method takes advantage of the availability of the higher derivatives in time for the em fields supported by the recently developed high order field solver which can also calculate the derivatives at the particle position. The kind of field solver which provides these derivates is independent from particle treatment module. On 3 examples we could give a first proof of the increased accuracy and effective order of convergence even in the relativistic case of the $\vec{E} \times \vec{B}$ drift. The relativistic cases were only in the region of a constant γ. The further tests should performed on cases were the derivates of γ exists. The investigation of the stability limits represents an essential point of interest too.

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