Theoretical Analysis of the Influence of the Power Supply on Breathing Oscillations in Hall Thrusters

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The breathing mode is the best known low frequency longitudinal bulk instability of Hall thrusters, capable of generating very wide, regular discharge current oscillations in the $10-30$ kHz range. This study extends a recent theory of breathing mode oscillations to the case of a non-ideal voltage source. A simple equivalent circuit modeling the AC behavior of the thruster is derived, using R, L and C components. The equivalent circuit explains in a straightforward way why an impedance in series with the generator is usually able to damp oscillations. More generally, the derived equivalent circuit can be expected to greatly improve the understanding of interactions between the thruster and the power processing unit, and in turn to help the design of robust filters.

Nomenclature

Roman symbols

A	Cross-section area of the channel
I	Discharge current
m	Atomic mass of the propellant
$\dot{m}$	Mass flow rate of propellant
n	Plasma density
n^*	Scaled plasma density
N	Density of neutral species
U	Discharge voltage
V	Axial velocity of neutrals

Greek symbols

β_i	Ionization rate
ν	Instantaneous relative growth rate of I
ν_{iw}	Ion-walls collisions frequency

I. Introduction

Although longitudinal low frequency oscillations have likely been observed in virtually all Hall thrusters since their inception, the term "breathing" oscillations was only coined in 1998 by Boeuf and Garrigues, who observed an apparent back-and-forth motion of the ionization front in a numerical simulation of the discharge at constant discharge voltage.[1] Ionization was also identified as a key element in theoretical models proposed shortly before by Baranov *et al* on the one hand,[2] and by Martinez-Sanchez and Fife on the other hand.[3]

The essential role played by ionization has not always been as obvious as it appears today. For early experimenters, the most striking feature of low frequency oscillations was their high sensitivity to power supply characteristics, attributed to a "circuit", "loop" or "contour" instability in the Russian literature.[4] The actual underlying physical phenomenon has for long been disregarded, while most efforts focused on devising filters that would limit the transient load on the power supply.[5,6] To this day, low frequency filters remain an essential element of the design of power processing units for Hall thruster.

An in-depth theoretical analysis of breathing oscillation in the case of constant discharge voltage has been recently proposed[7] and is still being developed.[8] It was found that due to the disparate transit-time scales of neutrals and charged species, the latter can be considered to be at any moment in quasi-equilibrium with the flow of neutral particles. The only transient equations left in such a

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model are the equations for the dynamics of neutrals, and a first order time-differential equation for the discharge current. This model was successfully validated against a numerical simulation, and was shown to constitute a closely related but more rigorous formulation of the predator-prey model of Fife and Martinez-Sanchez.[3]

Because Hall thrusters typically operate with a power supply coupled to a RLC filter, this study extends our previous work by accounting for a non-ideal voltage source.

II. Model based on quasi-steady dynamics for charged species

Following Ref. 7 the transport of neutrals along the main axis of the thruster is described by a simple advection equation, with a source term accounting for ionization and for the recombination of ions at the walls,

$$\frac{\partial N}{\partial t} + V\frac{\partial N}{\partial x} = -\beta N n + \nu_{iw} n, \tag{1}$$

$$N(0,t) = \frac{\dot{m}}{AVm} = \text{const}, \tag{2}$$

where N, n, β, ν_{iw} are functions of x and t. For the sake of simplicity, the velocity of neutrals, V, is assumed constant.

It was formerly noted[7] that the dynamics of neutrals is sufficiently slow to consider that the quantities relating to the fast dynamics of charged species, such as β and ν_{iw} and U, reach at any moment a quasi-steady state that solely depends on the instantaneous value of function N and on the relative growth rate of the discharge current, $d\ln I/dt$. Although the plasma density n constitutes an exception to this rule, it was found that its ratio to the discharge current does reach a steady state. Summing up, one formally writes,

$$\frac{n}{I} = \mathcal{S}_1\left(\frac{d\ln I}{dt}, N\right), \tag{3}$$

$$\beta = \mathcal{S}_2\left(\frac{d\ln I}{dt}, N\right), \tag{4}$$

$$\nu_{iw} = \mathcal{S}_3\left(\frac{d\ln I}{dt}, N\right), \tag{5}$$

$$U = \mathcal{U}\left(\frac{d\ln I}{dt}, N\right), \tag{6}$$

where the $\mathcal{S}_i$ are operators and $\mathcal{U}$ a functional, acting on the scalar $d\ln I/\mathrm{d}t$ and on function $N(x)$. In our previous work, U was considered constant so that the last relationship would simply provide an expression of $d\ln I/\mathrm{d}t$ as a function of N, which in turn allows other quantities to be expressed as functions of N only . In the present work, a non-ideal voltage source is accounted for, meaning that U must be treated as an external variable. Using Eqs (1–6), a compact form of the model of the thruster is obtained,

$$\frac{\partial N}{\partial t} + V\frac{\partial N}{\partial x} = -I\mathcal{S}(\nu, N), \tag{7}$$

$$N(0,t) = \text{const}, \tag{8}$$

$$\frac{d\ln I}{dt} = \nu, \tag{9}$$

$$U = \mathcal{U}(\nu, N). \tag{10}$$

Let us note that in order to obtain a closed system, these equations should be completed by a characteristic equation for the power processing unit (PPU) relating the behaviors of U and I.

III. Linear model

The linear counterpart of the above non-linear model can be derived after the introduction of small harmonic perturbations of N, I, ν and U around their steady state,

$$N(x,t) = \bar{N}(x) + \hat{N}(x)\exp(\jmath\omega t), \tag{11}$$

$$
\begin{aligned}
I(t) &= \bar{I} + \hat{I}\exp(\jmath\omega t), && (12)\\
\nu(t) &= \hat{\nu}\exp(\jmath\omega t), && (13)\\
U(t) &= \bar{U} + \hat{U}\exp(\jmath\omega t). && (14)
\end{aligned}
$$

The model is then linearized as follows,

$$
\begin{aligned}
\jmath\omega\hat{N} + V\frac{d\hat{N}}{dx} &= -\bar{S}\hat{I} - \bar{I}\left[S_\nu\hat{\nu} + \mathcal{S}_N\left(\hat{N}\right)\right], && (15)\\
\hat{N}(0) &= 0, && (16)\\
\jmath\omega\frac{\hat{I}}{\bar{I}} &= \hat{\nu}, && (17)\\
\hat{U} &= u_\nu\hat{\nu} + \mathcal{U}_N\left(\hat{N}\right). && (18)
\end{aligned}
$$

$\bar{S} \equiv \mathcal{S}\left(0, \bar{N}\right)$ is a function of x and corresponds that the steady state source term. S_ν is a function of x that arises from the differentiation of operator $\mathcal{S}$ with respect to ν. $\mathcal{S}_N$ is a linear operator obtained from differentiation of $\mathcal{S}$ with respect to N. u_ν is a scalar that stands for the derivative of $\mathcal{U}$ with respect to ν. $\mathcal{U}_N$ is in turn a linear functional obtained by differentiation of $\mathcal{U}$ with respect to N.

Defining operator $\mathcal{T}_N\left(\hat{N}\right) \equiv \bar{I}\mathcal{S}_N\left(\hat{N}\right) - \left(\bar{I}S_\nu/u_\nu\right)\mathcal{U}_N\left(\hat{N}\right) + V\left(d\hat{N}/dx\right)$, function $T_U(x) \equiv \bar{I}S_\nu(x)/u_\nu$ and the characteristic inductance of the thruster,

$$
L \equiv \frac{u_\nu}{\bar{I}}, \tag{19}
$$

the linear model can finally be expressed as,

$$
\begin{aligned}
\jmath\omega\hat{N} &= -\bar{S}\hat{I} - T_U\hat{U} - \mathcal{T}_N\left(\hat{N}\right) && (20)\\
\hat{N}(0) &= 0 && (21)\\
L\jmath\omega\hat{I} &= \hat{U} - \mathcal{U}_N\left(\hat{N}\right) && (22)
\end{aligned}
$$

IV. Approximate equivalent electrical circuit

A. Equivalent impedance of the thruster

Equivalent circuits can be instrumental in understanding the interactions between a device and its power supply. Although the linear model proposed above constitutes a great improvement in term of simplicity over classical models of Hall discharges, it is still too complex for the purpose of understanding PPU effects. Further simplifications can be achieved by resorting to an asymptotic approach based on a small parameter of the order of the ratio of densities, $\varepsilon = \mathcal{O}(n/N)$. The details of this asymptotic approach will be presented in a separate work.[8] In substance it can be shown that, provided an adequate dimensionless scaling, the inductance in the linear system can be treated as a small quantity of order ε. Subsequently, it is found that the frequency behaves as a large quantity of order $\varepsilon^{-1/2}$. Noting furthermore that the ion production at anode is very small, $\bar{S}(0) \approx 0$, it is ultimately found that $\hat{N}(x)$ is at the leading order a standing wave,

$$
\hat{N} = -\frac{1}{\jmath\omega}\bar{S}\hat{I} + \mathcal{O}\left(\frac{\hat{I}}{\omega^2}\right) \tag{23}
$$

Defining now the scalar,

$$
\hat{W} = \mathcal{U}_N\left(\hat{N}\right), \tag{24}
$$

which has the dimension of an electrical potential, a useful approximation of $\hat{N}$ as a function of $\hat{W}$ can be obtained,

$$
\hat{N} = \frac{\bar{S}\hat{W}}{\mathcal{U}_N\left(\bar{S}\right)} + \mathcal{O}\left(\frac{\hat{W}}{\omega}\right). \tag{25}
$$

The key in obtaining a simple constitutive equation from the linear model is to substitute the equation for $\hat{N}$, which is a function of x, by an equation for $\hat{W}$, which is a scalar quantity. By virtue of the asymptotic analysis, which states that $\mathcal{T}\left(\hat{N}\right)$ is only of order $\sqrt{\varepsilon}$ compared to $\bar{S}\hat{I}$, Eq. (20) can be simplified by accounting only for the leading order of Eq. (25) in estimating $\mathcal{T}\left(\hat{N}\right)$, leading to,

$$\jmath\omega\hat{N} = -\bar{S}\hat{I} - T_U\hat{U} - \frac{\hat{W}}{\mathcal{U}_N\left(\bar{S}\right)}\mathcal{T}_N\left(\bar{S}\right) + \hat{W}\mathcal{O}\left(\sqrt{\varepsilon}\right). \tag{26}$$

The linear functional $\mathcal{U}_N$ can be in turn composed with the former equation to obtain a scalar equation,

$$\jmath\omega\hat{W} = -\mathcal{U}_N\left(\bar{S}\right)\hat{I} + \mathcal{U}_N\left(T_U\right)\hat{U} + \frac{\mathcal{U}_N\left[\mathcal{T}_N\left(\bar{S}\right)\right]}{\mathcal{U}_N\left(\bar{S}\right)}\hat{W} + \hat{W}\mathcal{O}\left(\sqrt{\varepsilon}\right). \tag{27}$$

Defining appropriate conductances and capacitance,

$$G_U = \frac{\mathcal{U}_N\left(T_U\right)}{\mathcal{U}_N\left(\bar{S}\right)}, \tag{28}$$

$$G_W = \frac{\mathcal{U}_N\left[\mathcal{T}_N\left(\bar{S}\right)\right]}{\left[\mathcal{U}_N\left(\bar{S}\right)\right]^2}, \tag{29}$$

$$C_W = -\left[\mathcal{U}_N\left(\bar{S}\right)\right]^{-1}, \tag{30}$$

and neglecting the trailing order term, a simplified model is obtained,

$$C_W\jmath\omega\hat{W} = \hat{I} - G_U\hat{U} - G_W\hat{W}, \tag{31}$$

$$L\jmath\omega\hat{I} = \hat{U} - \hat{W}. \tag{32}$$

Eliminating $\hat{W}$, this model readily leads to the AC impedance of the thruster,

$$\frac{\hat{U}}{\hat{I}} = \frac{L\jmath\omega\left(C_W\jmath\omega + G_W\right) + 1}{C_W\jmath\omega + G_W + G_U}. \tag{33}$$

For an ideal voltage source ($\hat{U} = 0$), ω is given by,

$$\omega = \sqrt{\frac{1}{LC_W} - \frac{G_W^2}{4C_W^2}} + \jmath\frac{G_W}{2C_W}. \tag{34}$$

Recalling that L can be considered a small quantity, the oscillation frequency may be then approximated by,

$$\frac{1}{2\pi}\Re\left(\omega\right) \approx \frac{1}{2\pi\sqrt{LC_W}}, \tag{35}$$

while the growth rate is,

$$-\Im\left(\omega\right) = -\frac{G_W}{2C_W}. \tag{36}$$

Therefore, with an ideal voltage source, breathing oscillations can be expected whenever $G_W < 0$ (assuming $L > 0$ and $C_W > 0$).

B. Equivalent circuit

The translation of Eq. (33) into an equivalent circuit is not unique. Restricting our choice to R,L,C components, a relatively simple circuit can be devised by expressing the impedance as

$$\frac{\hat{U}}{\hat{I}} = \jmath\omega L + R + \frac{1}{C\jmath\omega + G} \tag{37}$$

where,

$$R = -\frac{LG_U}{C_W}, \tag{38}$$

$$G = \frac{G_U + G_W}{1 - R\left(G_U + G_W\right)}, \tag{39}$$

$$C = \frac{C_W}{1 - R\left(G_U + G_W\right)}. \tag{40}$$

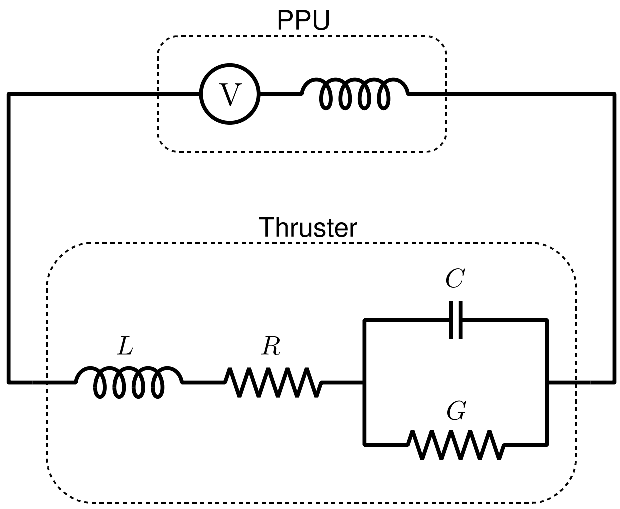

Figure 1. Equivalent circuit of the thruster in series with a simple PPU.

The equivalent circuit is depicted on Fig. 1.

We readily note that R is proportional to L and is thus a small parameter. Consequently, we have $G \approx G_W + G_U$ and $C \approx C_W$. In term of the new circuit components, the frequency of the mode is thus approximately

$$\frac{1}{2\pi}\Re(\omega) \approx \frac{1}{2\pi\sqrt{LC}}, \tag{41}$$

and the growth rate is,

$$-\Im(\omega) = -\frac{G}{2C} - \frac{R}{2L}. \tag{42}$$

For illustration purposes, Fig. 2 shows the values of L and C at various operating voltages extracted from a time-dependant non-linear 1D numerical model of the discharge,[7,8] using Eqs (19, 40). The frequency from Eq. (41) was also compared in Fig. 3 to the actual frequency obtained by processing the simulated discharge current time series, showing a very good agreement. Likewise, the region where $G_W < 0$ in Fig. 4 recovers nearly exactly the threshold voltage beyond which spontaneous oscillations are observed in simulations with an ideal voltage source.

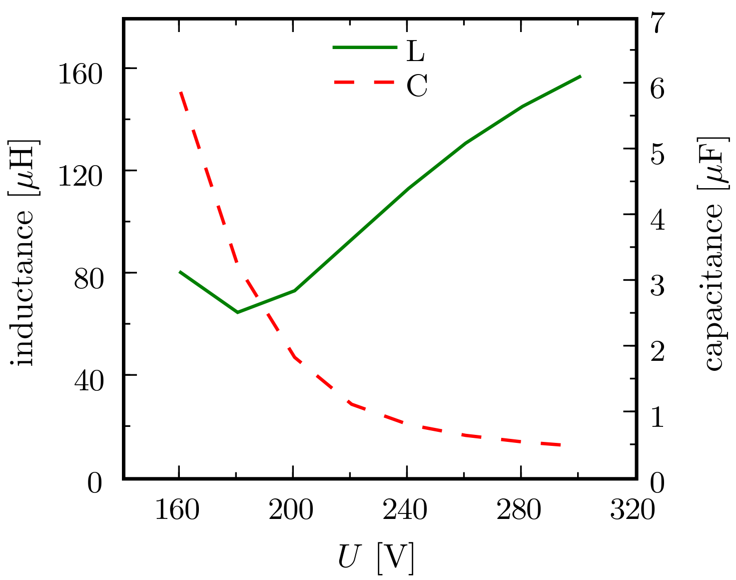

Figure 2. L and C as a function of U extracted from a numerical simulation of the discharge.[7,8]

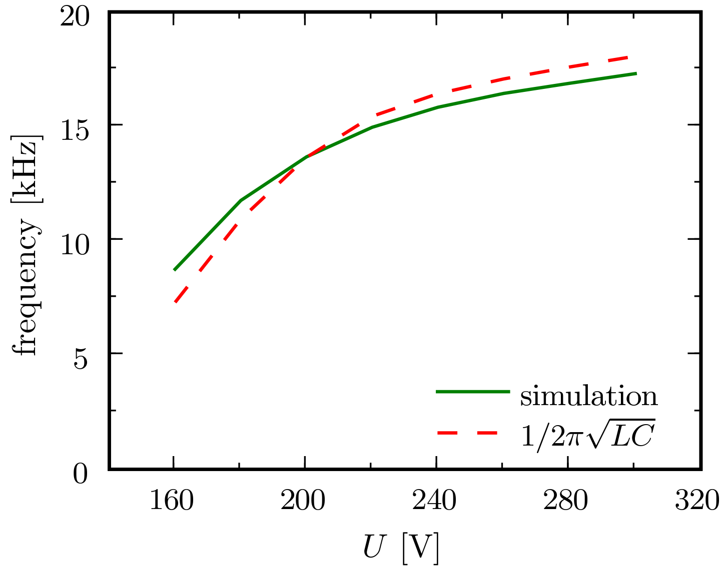

Figure 3. Frequency as a function of U as observed in a numerical simulation of the discharge,[7,8] compared to the prediction of the equivalent circuit.

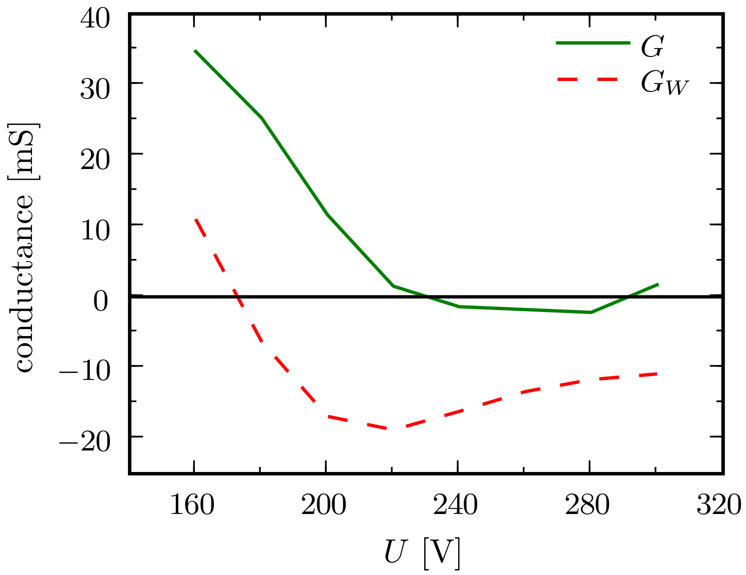

Figure 4. characteristic conductances of the equivalent circuit extracted from a numerical simulation of the discharge.[7,8] Regions with $G_W < 0$ are expected to be unstable. Inductor-only filters are expected to be efficient where $G > 0$.

C. Example of application: inductor or resistor as a PPU filter

Although modern PPUs are somewhat more complex, inductor-only filters have been commonly used for many years in experimental setups. The popularity of inductor-only filters is easily explained by their simplicity and the (theoretical) non-dissipative nature of inductors.

The fact that a single inductor may be sufficient to damp oscillations can be understood by analyzing the complete circuit of Fig. 1. It is readily seen that the two inductors are in series, so that the circuit behaves exactly as a thruster with impedance $L + L_{PPU}$powered by an ideal voltage source. The growth rate for the full system is then inferred from Eq. (42) as,

$$-\Im(\omega) = -\frac{G}{2C} - \frac{R}{2(L + L_{PPU})}. \tag{43}$$

Provided that $G > 0$, it is thus possible to recover stability with a sufficiently large inductor in series with the PPU.

In the situation when $G < 0$, stability can still be achieved using a resistor in series with the generator. Using the same argumentation as above, it is easily obtained that the growth rate becomes then,

$$-\Im(\omega) = -\frac{G}{2C} - \frac{R}{2L} - \frac{R_{PPU}}{2}. \tag{44}$$

The later strategy is obviously much less interesting as it leads to power dissipation inside the PPU.

V. Conclusion

A method has been outlined to obtain an equivalent circuit meant to model the low-frequency AC behavior of a Hall thruster. A short analysis of the case of a single inductor or resistor in series with the generator has been carried out, illustrating the perspectives offered by this method. Also, the ability of the model to recover the breathing frequency and the stability threshold with good accuracy has been demonstrated on the basis of numerical simulations.

The validation and exploitation of this method shall be the object of further works. In particular, additional parametric simulations may allow to identify the regions where G is typically negative (that is, where an inductor-only filter become inefficient) and to devise appropriate PPU filters.

In a farther perspective, experimental works might be performed to verify the expected effect of PPU filters on the mode frequency and stability.

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